Endogenous Timing in a Quantity Setting Duopoly

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Abstract

I provide an equilibrium analysis of the role of private information on timing decisions of duopolists in a quantity setting framework. The firms are privately informed about their costs and may decide their quantities in one of two dates. In the unique symmetric perfect Bayesian equilibrium, a firm with a low cost produces in the first date, while a firm with a high cost produces in the second date, after having learned the other firm’s decision in the first date. Thus, if the firms have different costs, a leader-follower outcome will emerge, with leadership by the low cost firm. If the firms have the same cost, they will either play a Cournot outcome if costs are high, or both will attempt to lead in the market if both costs are low.

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1 Introduction

The Cournot and the Stackelberg models are certainly the most used duopoly models. They are the first duopoly models taught at the undergraduate level. They are used as simple applications of the Nash equilibrium in static games of complete information and the backwards induction equilibrium in dynamic games of complete information. They are the building blocks of most of the theoretical, applied and empirical industrial organization. Nevertheless, we have a limited understanding about the fundamental determinants that may lead to sequential decisions in some markets and simultaneous decisions in other markets. How is that the firms come about playing one game or the other? How do they learn to play the equilibrium strategies? If a firm in an homogenous good duopoly is better off by leading, why is that the many real life markets are reasonably well described by the Cournot model?

In most economic analysis of markets, the selection between the Cournot and the Stackelberg outcomes is exogenously motivated. It follows from a given timing structure of the game, that is postulated as a primitive of the model in a way that seems plausible and adequate to study the issue at interest. However, there are situations in which the choice of the timing is the fundamental decision. For example, the opening of a new market, with the firms having to decide when to enter, creates a natural framework for the study of the selection between sequential moves and simultaneous moves. What are the trade-offs between entering in an early date and waiting to enter later? What outcome should one expect? This paper attempts to contribute for our understanding of the answer to such questions.

There is now a large body of literature around this general issue. A first group of papers studies the preferences of the firms over the roles of leader or follower in duopolies. For example, Gal-Or (1985), Boyer and Moreaux (1987) and Dowrick (1986) showed that the firms’ preferences over the two roles depend on the slope of the reaction curves, which is very much related to the space in which competition takes place. In a subsequent paper, Gal-Or (1987), considering a model with an incumbent firm that has superior information about a parameter of the demand than an entrant firm, concluded that the state of the demand affects the attractiveness of being a leader. Albæk (1990) contributed to this literature by considering a model in which the firms commit to a timing of production (i.e., choose between sequential and simultaneous moves) before knowing their costs. If the difference between the variance of the firms’ costs is sufficiently large, there will be Stackelberg
leadership of the firm with the greater variance. This first generation of papers is not as much about the choice of the timing of the game as it is about the selection between alternative extensive form games, based on comparisons of profits across equilibria.

Endogenous timing decisions, modelled through games in which the firms choose when to produce, have been studied in a variety of settings. Hamilton and Slutsky (1990), considering extensive form games of complete information with varying degrees of commitment, concluded that, the Stackelberg outcome (with sequential play) emerges as the equilibrium. Mailath (1993), building on the model of Gal-Or (1987), considered an incomplete information game, where there is an informed firm (incumbent) and an uninformed firm (entrant). The informed firm had the option of either choosing its quantity before the uninformed firm or delay until the decision period of the uninformed firm. Is was shown that the unique stable sequential equilibrium is fully revealing, with the incumbent acting as the Stackelberg leader. Normann (1997) generalized this model letting either firm commit to a production in any of the two dates, concluding that Stackelberg equilibria, with either the informed or the uninformed firm moving first, emerge endogenously. More recently, Normann (2002) analyzed endogenous timing, as in the extended game with observable delay of Hamilton and Slutsky (1990), on the basic market model of Mailath (1993). Stackelberg equilibria, with either the informed or the uninformed firm as the leader, may endogenously emerge, but so does as well, for most parameter values, the Cournot equilibrium in the first period.

Endogenous Stackelberg outcomes also emerge in other models. Maggi (1996) considers strategic investment under uncertainty in a new market, where firms face a tradeoff between commitment and flexibility. Under fairly general conditions, the subgame perfect equilibrium is asymmetric, with one firm committing to early investment and the other firm waiting and investing later. Moreover, as the uncertainty vanishes, the model predicts convergence to the full information Stackelberg outcome. A two-stage game in which each player can either commit to a quantity in stage 1 or stage 2 is considered in van Damme and Hurkens (1999). It is shown that committing is more risky for the high cost firm, and so risk dominance considerations lead to the conclusion that only the low cost firm will choose to commit. Hence, the low cost firm will emerge as the Stackelberg leader. However, this procedure does not select the Cournot outcome, if the firms have the same cost.

Although asymmetric Stackelberg outcomes are fairly consistent in all these papers, some modeling approaches have yielded predictions compatible with Cournot outcomes. Saloner (1987) as-
sumed that the firms could produce at two dates before the market cleared, with the choices of the first date being observed before the choices of the second date had been made. He concluded that, when the cost of production does not change from the first to the second date, a multiplicity of subgame perfect equilibria exist, which include the Cournot and the Stackelberg outcomes. Pal (1991) studied the equilibria when the production costs change between the dates. If the costs fall slightly from the first to the second date, there are multiple leader-follower equilibria; otherwise, in equilibrium the firms will produce the Cournot quantities in the date in which production is cheaper. When costs decrease slightly over time, there is also a mixed strategy equilibrium, in which each firm randomizes over producing in one date or the other (Pal, 1996). Romano and Yildirim (2002) analyze games in which agents can adjust their strategic variable upwards over a sequence of periods, until the market clears. In the case of a duopoly they show that the Cournot and the Stackelberg outcomes can emerge in equilibrium. At a general level, Amir and Grilo (1999) provide conditions on demand and cost functions of a quantity setting duopoly for Cournot or Stackelberg outcomes to occur in equilibrium. A \( n \)-firm, \( m \)-period model, where each firm chooses both how much to produce and when to produce it, is formulated in Matsumura (1999). It is found that at least \( n - 1 \) firms simultaneously produce in the first period in every pure strategy equilibrium, showing that the generalized Stackelberg-type outcome in a duopoly only. Finally, experimental evidence does not confirm the theoretical prediction of the emergence of endogenous Stackelberg leadership, as Cournot outcomes are often observed (Huck et al., 2002 and Fonseca et al., 2004).

The prediction of asymmetric equilibria with Stackelberg like outcomes is clearly the most frequent result in this literature. However, it is not clear that most of these papers provide theories of endogenous leadership. In one way, it is no surprise that Stackelberg outcomes, in which one firm leads no matter what and the other firm follows, will appear in equilibrium in most of these models; but these equilibria are not endogenously motivated by the primitives of the models. Moreover, in models with asymmetrically informed firms (e.g., Mailath, 1993), the equilibria Stackelberg outcome seem to be driven by the asymmetry. On the other way, the mixed strategy equilibrium in Pal (1996) suggests that a model of \textit{ex-ante} symmetric firms, which get private information on a relevant parameter, might yield a more general and possibly different prediction. This is the main motivation for the present paper.
The periods before starting production in a new market should involve a lot of uncertainty. In this paper I want to concentrate on the role of uncertainty about costs. If there are several technologies that might be chosen by the potential duopolists, it is plausible that in many situations the firms will have private information about their costs. I explore the role of this asymmetry to explain different decisions relative to the moment of production. Like Mailath (1993), I construct a game of incomplete information. However, unlike Mailath (1993), the firms are \textit{ex-ante} identical. They may choose to produce at date 1 or to wait and see the other’s date 1 decision and produce at date 2 only. Considering a model with two types of firms and a linear demand, I show that in the unique symmetric time-dependent equilibrium, a firm with low costs will produce at date 1 (possibly randomizing over production at date 2), while a firm with high costs will produce at date 2 (possibly randomizing with production at date 1). This analysis provides an endogenous theory for the timing of production in a duopoly.

The results in this paper may help interpret the results in some other papers. In Pal (1996) the author identified a mixed strategy equilibrium that yields the same kind of outcomes as the time-dependent equilibrium in this paper. This result, may suggest models in other papers (e.g., Maggi, 1996) may also have mixed strategy equilibria with similar outcomes. In such a case, the endogenous leadership results in some papers could rely on the restriction of the analysis to pure strategy equilibria.

A result similar to the central proposition in this paper is provided in van Damme and Hurkens (1999). They were the first to obtain the result that if the duopolists have different costs, the low cost firm will act as the leader. But, there is a fundamental difference between the two papers. Their result is derived from a game theoretic argument of risk dominance applied to the pure strategy equilibria of a complete information game, while I identify the equilibria in an incomplete information game. On the other hand, the two papers provide different predictions in the case in which the two firms have the same cost.

This paper is organized in the following way. In section 2 I describe the basic structure of the model analyzed in the paper and introduce some notation. Then, in section 3, I provide an equilibrium analysis of the model with two types of firms. Section 4 is devoted to the discussion of some extensions of the model. Conclusions are presented in section 5. The proofs are collected in an Appendix.
2 The Model

2.1 The Elements of the Model

There are two firms, which may produce a homogeneous good, with total demand described by $Q^D(p) = a - p$. The firms produce with constant marginal costs and have no fixed costs. The marginal costs are privately known and independently drawn from $C$, according to some cumulative distribution function, $F(\cdot)$. Moreover, it is implicitly assumed that the parameters are such that the market will never be monopolized.

2.2 The Timing of the Decisions

The firms must decide a quantity to be produced at one of two dates. At date 1 they simultaneously decide how much to produce. These decisions are then publicly revealed. Any firm that does not produce at date 1, may decide its level of production at date 2. Finally, at date 3, given the productions decided, the market clears. The timing of the game is depicted in figure 1.

For a firm there is a clear trade-off between the timing decisions. A firm that produces at date 1 may get the benefit of acting as a leader, producing first and influencing the other firm’s production, if the latter has decided to produce at date 2. However, by producing at date 1, a firm decides without knowing anything about the other firm’s decision, and there is the possibility that the other firm also produces at date 1. To the contrary, a firm that decides to wait, cannot take the role of a leader, but will have more information when deciding. Thus, the structure of the model provides a framework to study the extent to which the cost level of a firm might influence the moment of production in a quantity setting duopoly, when firms have private information about their costs.

The choices of the firms regarding the moment of production can be described in terms of
the leader-follower dichotomy: a firm that produces at date 1 acts as a leader, while a firm that produces at date 2 acts as a follower. Several outcomes are therefore possible: a Cournot outcome will result, if both firms wait to produce at the date 2; a leader-follower outcome (in the spirit of Stackelberg) will emerge, if one firm produces at date 1 and the other does it at date 2; and a double leadership outcome appears if both firms produce at date 1.

2.3 Some Notation

Before proceeding with the equilibrium analysis, some notation is introduced. A pure strategy for firm $i$, with $i \in \{1, 2\}$, is described by $s_i = (\tau_i, \lambda_i, \phi_i, \xi_i)$, where $\tau_i : C \rightarrow \{1, 2\}$ is the moment of production, $\lambda_i : C \rightarrow \mathbb{R}_+$ is the quantity chosen if producing at date 1, $\phi_i : \mathbb{R}_+ \times C \rightarrow \mathbb{R}_+$ is the quantity chosen if producing at date 2, after the other firm has produced at date 1, and $\xi_i : C \rightarrow \mathbb{R}_+$ is the quantity chosen if producing at date 2, when the other firm also produces at date 2.

The production level of firm $i$, with $i \in \{1, 2\}$, is denoted by $q_i$, and the ex-post profit function of firm $i$ by $\pi_i(q_i, q_{3-i}, c_i)$. This profit function is continuous, concave in the own production and strictly decreasing in the other firm’s production and the own cost. Therefore, once having decided the timing of production, no firm will randomize over the production quantities. Hence, the only mixed strategies that need to be considered are those in which a firm randomizes over the moment of production. Hence, one needs only to consider mixed strategies $\sigma_i = (\alpha_i, \lambda_i, \phi_i, \xi_i)$, where $\alpha_i : C \rightarrow [0, 1]$ denotes the probability that firm $i$ will produce at moment 1.

Given a strategy of firm $i$, $\beta_i$, with $i \in \{1, 2\}$, will be used to denote the ex-ante probability that firm $i$ will produce at date 1, i.e.,

$$\beta_i = \int_C \alpha_i(c) dF(c);$$

and $\Pi^t_i(c_i, \sigma_{3-i})$ will denote the maximum expected ex-ante profit of firm $i$, with cost $c_i$, if it produces at date $t$ and the other firm uses the strategy $\sigma_{3-i}$.

3 Equilibria in the Two Types Model

In this subsection I analyze the two types version of the model. It is assumed that the marginal costs are privately known and independently drawn from $C = \{c, \bar{c}\}$, with $a > \bar{c} > c$, and $\gamma \in (0, 1)$.
is the probability of drawing the low cost. For the equilibrium analysis I use the concept of Perfect Bayesian equilibrium.

3.1 The Stackelberg Time-independent Equilibria

Even though the firms have private information about their costs it is possible that such information is ignored when firms decide the moment of their production.

Proposition 1 There are two pure strategy equilibria in which one of the firms produces at date 1, independently of its own cost, and the other firm produces at date 2, independently of its own cost.

These equilibria are a generalization of the standard Stackelberg equilibria to this model of a duopoly with incomplete information on firms costs. They tend to be present in most models that allow for an endogenous choice of the moment of production. Moreover, Stackelberg equilibria of have been predicted as the outcome in most of the literature referred in the Introduction.

In these Stackelberg equilibria, the firms’ timing decisions are independent of their costs. The timing of production does not provide any information about the firms’ costs. Their presence in this model raises an issue of multiplicity of equilibria. Possibly, these equilibria could be ruled out by carefully crafted change in the modelling assumptions. However, as the main objective of this paper is to understand how private information about the costs of production may impact the timing of production, I decided to keep this simple model and, from now, I concentrate on the characterization of time-dependent equilibria.

3.2 Time-dependent Equilibria

In this subsection I look at and characterize equilibria in which the firms choose the timing of their production contingent on their costs.

A key result for the analysis is provided in the following proposition.

Proposition 2 There is a unique time-dependent equilibrium. In this equilibrium, which is symmetric, a firm with a high cost does not produce earlier than a firm with a low cost.

The basic intuition behind this result is that, under similar conditions, the impact on profit of anticipating production to the first moment is larger for a firm with a low cost than for a firm with
a high cost. Therefore, whenever a firm with a high cost has an incentive to set production on the first date, so does a firm with a low cost. But the converse is not true.

As a consequence of the proposition one may conclude that, in a time-dependent equilibrium, a firm with a low cost will produce at date 1 with positive probability and a firm with a high cost will produce at date 2 with positive probability (and they will never both randomize over the moment of production). Hence, in a time-dependent equilibrium, it is not possible to observe a high cost firm leading and a low cost firm following. This is the first main conclusion of this analysis.

The remaining of the subsection provides a deeper characterization of the time-dependent equilibria. The structure of the equilibria depends on the values of the parameters. Two cases need to be considered.

3.2.1 Equilibrium with High Cost Differences

The case in which \( \bar{c} - c \) is sufficiently high is considered first. Under this condition, in equilibrium, the moment of production fully reveals the firm’s cost. The equilibrium behaviors are described in the following proposition.

**Proposition 3** If the difference in the two possible costs is sufficiently high, i.e. \( (\bar{c} - c)/(a - c) \) is above a certain cutoff value that depends on \( \gamma \), in the unique time-dependent equilibrium, a firm with a low cost will produce at date 1, while a firm with a high cost will produce at date 2.

This proposition is a clear illustration of the main result of this paper. If the firms play the time-dependent equilibrium, a firm with a low cost will lead, by deciding to produce at date 1, and a firm with a high cost will wait and produce at date 2, behaving as a Stackelberg follower, if the other firm has produced at date 1, or as a Cournot duopolist, if the other firm has also waited.

Thus, equilibrium outcomes of the game are the following:

- If both firms have high costs, they will both produce the Cournot outcome at date 2;
- If one firm has a low cost and the other has a high cost, the former will lead and the latter will follow, thus providing a Stackelberg like leader-follower equilibrium outcome;
- If both firms have low costs, they will both produce at date 1, attempting to be leaders.
3.2.2 Equilibrium with Low Cost Differences

The strategies described in Proposition 3 are an equilibrium if and only if $\bar{c} - c$ is sufficiently high. If the difference in costs is lower, the previous strategies will not be an equilibrium. A firm with a high cost would gain by deviating and produce at date 1. However, it cannot be an equilibrium that both firms produce at date 1. Hence to obtain an equilibrium, it has to be the case that a firm with cost equal to $c$ will still produce at date 1, but a firm with cost equal to $\bar{c}$ will randomize between producing at date 1 or waiting to produce at the date 2. These equilibrium behavior is summarized in the following proposition.

**Proposition 4** If the difference in the two possible costs is not sufficiently high, i.e. $(\bar{c} - c)/(a - c)$ is below a certain cutoff value that depends on $\gamma$, in the unique time-dependent equilibrium, a firm with a low cost will produce at date 1, while a firm with a high cost will randomize over the date of production.

Therefore, below the cutoff value for $\bar{c} - c$ the moment of production will not fully separate the firms. A firm with cost equal to $c$ will always produce at date 1, while a firm with cost equal to $\bar{c}$ will randomize between producing at date 1 and producing at date 2. Nevertheless, the quantities produced at date 1 allow for the separation of the two types of firms.

3.3 Comments on the Equilibria

So, three equilibria of the game analyzed in this section exist: the two time-independent Stackelberg equilibria and a symmetric (separating) time-dependent equilibrium, in which a firm with a low cost produces at date 1, while a firm with a high cost produces at date 2 (possibly randomizing with production at date 1, if the two costs are not too different). The following picture provides a view on how the parameters affect the characteristics of the time-dependent equilibrium.

Hence, for a large set of parameter values the equilibrium is the one characterized in Proposition 3, with the firms separating on the moment of production depending on its cost.\(^1\)

This model provides a rational to support the argument, that if the duopolists select the equilibrium on the basis of cost asymmetries, one should observe a low cost firm trying to lead while a

\(^1\)It should be noted, however, that for the analysis in this paper only applies for $(\bar{c} - c)/(a - c) \leq (1+3\gamma)/(3(1+\gamma))$. Otherwise, there would be situations in which the market could be monopolized.
high cost firm accepts to follow. Why is there such a result, and no other equilibria was identified?

If there were no asymmetric information any firm would prefer to lead, committing to production at date 1. Leadership is the most profitable behavior when firms choose (strategic substitute) quantities. However, with asymmetric information there is a clear trade-off: leading, through production at date 1, has the usual advantages of leadership, but it opens up the unprofitable possibility that the other firm will also try to lead; on the other hand, following by delaying production to date 2, while giving up the benefits of leadership ensures that, when deciding, the firm will either know the other firm’s quantity or have a refined forecast. The firms trade-off these two effects. The result is this paper is that the difference in the profit from producing at date 1 and the profit from producing at date 2, is larger for a firm with a low cost than for a firm with a high cost. Hence, a firm with a low cost has a greater incentive to produce at date 1 than a firm with a high cost.

The fact that, for no parameter values, it is ever the case that, in the time-dependent equilibrium, a firm with a low cost will randomizes over the two moments of production and the firm with a high cost will produce at date 2 only, may deserve some additional explanation. Indeed, such possibility is not ruled out by Proposition 2. Moreover, one may wonder why is it that when the
probability of drawing the low cost decreases approaching 0, eventually a firm with a high cost will start randomizing over the moment of production, but as the probability increases approaching 1, it will never be the case that a firm with a low cost will want to randomize over the moment of production. In fact, if $\gamma$ approaches 1 it will be very likely that both firms will attempt to become leaders by producing in date 1, and hence ex-post regretting the decision. However, in dealing with this risk, a firm with a low cost prefers to adjust the quantity that it produces in date 1 instead of considering producing in date 2. In the limit, a firm with a low cost will produce the Cournot quantity in the date 1.

Highlighting the discussion in the previous argument, and as the game in this section is a game of incomplete information while most of the papers mentioned in the Introduction consider games of complete information, it may be useful to look at the limit of the symmetric time-dependent equilibrium identified in this section, as the asymmetry of information vanishes.\(^2\) One gets the following result:

**Corollary 4.1** In the limit, as the asymmetry of information vanishes, each firm will only produce on date 1 selecting a quantity equal to the Cournot outcome; these moments of production and quantities produced coincide with those in a subgame perfect equilibrium of the (limiting) game of complete information.

It should be noted that, like in the incomplete information game, the (limit) complete information game has three subgame perfect equilibria: the two Stackelberg equilibria and an equilibrium in which both firms produce the Cournot outcome at the date 1.

### 4 Some Extensions

There are some directions in which the analysis in the previous section may be extended. In this section the following extensions are considered: a continuum of types, existence of strategic complementarities, introduction of correlation on costs, allowing for production in both dates and the consideration of more general demand and cost functions.

\(^2\)The asymmetry of information may vanish in several ways, but they all yield the same limiting equilibrium productions.
4.1 A Continuum of Types

A possible generalization of the model is the extension to a continuum of types. So, in this section, we assume that the firms have costs continuously drawn from \( \mathcal{C} = [\bar{c}, \bar{c}] \), according to a cumulative distribution function that has no atoms. The analysis is restricted to time-dependent equilibria.

The main propositions about the time-independent equilibrium are straightforwardly generalized to the continuous cost case. They directly lead to the following proposition.

**Proposition 5** In the unique time-dependent equilibrium, which is symmetric, there exists a cost \( c^* \in (\bar{c}, \bar{c}) \) such that a firm with a lower cost \( (c < c^*) \) sets its production on date 1 only, while a firm with a higher cost \( (c > c^*) \) sets its production on date 2 only.

This Proposition sharpens the main result of the paper. It shows that the existence of equilibria in which a firm randomizes over the moment of production is a feature of the two types case. On the other hand, the property that leaders are low cost firm and follower are high cost firm is maintained.

4.2 Differentiated Goods

The basic model is that of a duopoly of a homogenous good. The firms produce a homogeneous good. In this subsection the analysis is extended to allow for differentiated goods, accommodating the cases of imperfect substitutability or complementarity. This is done by considering the following inverse demand functions:

\[
P_i^D(q_i, q_{3-i}) = a - q_i + \theta q_{3-i}, \quad \text{with } i \in \{1, 2\} \text{ and } \theta \in [-1, 1] \setminus \{0\}.
\]

The sign of \( \theta \) determines the strategic relationship between the products: if \( \theta \) is negative, the products are strategic substitutes; if \( \theta \) is positive, the products are strategic complements.\(^3\)

The equilibrium analysis in this case is very similar to that of the general model. Indeed, if the potential cost differences are sufficiently high, a firm with a low cost will prefer to produce at date 1 while a firm with a high cost will prefer to produce at date 2. Otherwise, a firm with a high cost will randomize over the two dates of production.

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\(^3\)If \( \theta = 0 \) the goods are independent and each firm becomes a monopolist in its own market. This leads to a degenerate case, to which most of the results derived in this subsection do not apply. However, for the sake of simplicity, this is ignored throughout the section.
The result in Proposition 2 for the homogeneous good model holds in this model with differentiated goods.

**Proposition 6** In the model with differentiated goods, there is a unique time-dependent equilibrium. In this equilibrium, which is symmetric, a firm with a high cost does not produce earlier than a firm with a low cost.

In the following proposition I provide the characterization of the equilibrium for high cost differences.

**Proposition 7** If the difference in the two possible costs is sufficiently high, i.e. \((\bar{c} - c)/\bar{a} - c\) is above a certain cutoff value that depends on \(\gamma\) and \(\theta\), in the unique time-dependent equilibrium, a firm with a low cost will produce at date 1, while a firm with a high cost will produce at date 2.

This proposition shows that, when the difference between the high cost and the low cost is sufficiently large, the properties of the time-dependent equilibrium identified in the homogeneous good model generalize to a model of differentiated goods, that allows for strategic substitutability or strategic complementarity.

If the difference between the high cost and the low cost is below a certain threshold, the following proposition will apply:

**Proposition 8** If the difference in the two possible costs is not sufficiently high, i.e. \((\bar{c} - c)/\bar{a} - c\) is below a certain cutoff value that depends on \(\gamma\) and \(\theta\), in the unique time-dependent equilibrium, a firm with a low cost will produce at date 1, while a firm with a high cost will randomize over the date of production.

Figure 3 provides a representation of the regions in the parameters space in which Proposition 7 and Proposition 8 apply. The several lines correspond to the frontier of the two regions for different values of \(\theta\): above the line Proposition 7 applies, while Proposition 8 applies below the line.

It is clear that when the goods are substitutes \((\theta < 0)\) it is easier to find parameter values for which the firms will be fully separated based on the moment of production. Although it can not be clearly inferred from the picture, it can be noted that as a negative \(\theta\) approaches 0, the line separating the two regions approaches the two axis. Then, when the goods become complements
and the degree of complementarity increases, the region of full separation based on the moment of production is fastly reduced.

4.3 Correlation on Costs

The analysis in the paper has been done under the assumption that the costs were independently drawn. In this subsection I explore the case of correlated costs.

Consider again the model with homogeneous goods. The costs are drawn from \( \{c, \bar{c}\}\), but are correlated, so that \( \text{Prob}\{c_{3-i} = c_i|c_i\} = (1 + \rho)/2 \), with \( i \in \{1,2\} \), where \( \rho \) is the coefficient of correlation among firm’s costs. Hence, if \( \rho = -1 \), the firms have different costs for sure; if \( \rho = 1 \), the firms have the same cost for sure; if \( \rho = 0 \), the costs are independent and equally likely.

This is obviously not a general treatment of the model with correlated costs. In particular, while maintaining a certain symmetry, so that symmetric equilibria may exist, this parameterization restricts the unconditional probabilities to be \( \text{Prob}\{c_i = c\} = \text{Prob}\{c_i = \bar{c}\} = 0.5 \).

Even within this parameterization of the model, not all results of the independent costs case extend to the correlated cost case. For example, if the correlation coefficient approaches \(-1\), so that the firms are almost sure to have different costs, there will be an equilibrium in which the firm
with the high cost produces at date 1 and a firm with a low cost produces at date 2. Hence, a result such as the one derived in Proposition 2 for the independent costs case, no longer holds for some parameter values when costs are correlated (see Proposition 11 below). However, the analysis in this subsection suggests that the main results about time-contingent equilibria still hold for most parameter values and, in particular, for the case of positive correlation (see Propositions 9 and 10 below).

The first proposition provides a characterization an equilibrium in which a firm with a low cost produces in date and a firm with a high cost produces in date 2, under a condition of high cost differences.

**Proposition 9** If the difference in the two possible costs is sufficiently high, i.e., \((\bar{c} - c)/(a - c)\) above a certain cutoff value that depends on \(\rho\), there is a time-dependent equilibrium in which a firm with a low cost produces at date 1 and a firm with a high cost produces at date 2. This equilibrium is unique and symmetric.

This is a generalization of Proposition 3. For positive and small negative values of \(\rho\), the impact of the condition is really similar to the one in the base model. If \(\rho\) becomes more negative, the equilibrium exists for all values of the parameters.

The equilibrium described in Proposition 9 does not exist, for some parameter values, because a firm with a high cost will prefer to also produce at date 1. The following proposition characterizes an equilibrium in such cases, generalizing the result in Proposition 4.

**Proposition 10** If the difference in the two possible costs is not sufficiently high, i.e. \((\bar{c} - c)/(a - c)\) is below a certain cutoff value that depends on \(\rho\), in the unique time-dependent equilibrium, a firm with a low cost will produce at date 1, while a firm with a high cost will randomize over the date of production.

However, these equilibria, which correspond to the unique time-dependent equilibria in the base model, do not exhaust the time-dependent equilibria if costs are correlated. For a small set of parameter values another symmetric equilibrium exists, as described in the following proposition.

**Proposition 11** If the two possible costs are very close, and the costs of the firms are highly negatively correlated, there exists a time-dependent equilibrium in which a firm with a low cost
produces at date 2 and a firm with a high cost produces at date 1. This equilibrium is unique and symmetric.

4.4 Production in both Dates

In the analysis so far firms have been restricted to produce in a single date. Hence, a firm that produces at date 1 is not allowed to produce more at date 2. One may, however, be interested in assessing the impact of such assumption on the properties of the equilibria identified.

Allowing for production in both dates raises signaling issues. A complete analysis of such a model deserve a paper by itself. Here, an example is provided. The example suggests that if firms may produce in both dates there may exist equilibria with properties that are different from the ones identified in the model with production in a single date.

Consider the following parameterization of the base model: \( a = 10, \bar{c} = 0, \bar{\bar{c}} = 2 \) and \( \gamma = 0.5 \). If there were full information about the firms costs and firms produced simultaneously, one would get the following outcomes:

- If both firms have costs equal to 0, both firms will produce \( \frac{10}{3} \);
- If one firm has cost equal to 0 and the other firm has cost equal to 2, the low cost firm will produce 4 and the high cost firm will produce 2;
- If both firms have costs equal to 2, both firms will produce \( \frac{8}{3} \).

Let us consider now the example with asymmetric information. Possibly, as in most signaling games, there are multiple equilibria. A full equilibrium analysis of this example is not provided here. However, the following result provides one equilibrium of this example, with features that are very different from those of the equilibria in the base model.

**Result:** There is a Perfect Bayesian equilibrium in which the firms produce according to the following strategy:

1. If the firm has cost equal to 0:
   - It will produce \( \frac{10}{3} \) at date 1;
   - If the other firm produced \( q \leq 2 \) at date 1, it will produce \( \frac{2}{3} \) at date 2;
• If the other firm produced \(2 < q \leq \frac{10}{3}\) at date 1, it will produce \(\frac{5}{3} - \frac{q}{2}\) at date 2;

• If the other firm has produced at least \(10/3\) at date 1, it will not produce at date 2.

2. If the firm has cost equal to 2:

• It will produce \(\frac{11}{4}\) at date 1;

• It will not produce at date 2.

After date 1, the firms will hold the following beliefs:

• If the other firm has produced less than \(\frac{10}{3}\) at date 1, it is believed to have cost equal to 2;

• If the other firm has produced at least \(10/3\) at date 1, it is believed to have cost equal to 0.

The differences between the behavior described in this equilibrium and the one described in the main analysis in this paper are clear.

In this equilibrium, both firms will produce at date 1 and only firms with the low cost produce at date 2 with positive probability. This is in contrast with the main result in this paper, that a firm with a low cost does not produce after a firm with a high cost.

Moreover, if the firms behave as described in the equilibrium, three outcomes may arise:

1. If both firms have a low cost, they will both produce at date 1, only; the quantities produced are those of the Cournot equilibrium of the full information game;

2. If one firm has a low cost and the other has a high cost, the firm with the low cost will produce \(\frac{10}{3}\) at date 1 and \(\frac{7}{27}\) at date 2, and the firm with a high cost will produce \(\frac{11}{4}\) at date 1, only; this outcome has features of a Stackelberg equilibrium with leadership of a high cost firm;

3. If both firms have a high cost, they will both produce \(\frac{11}{4}\) at date 1, only, involving overproduction compared to the full information equilibrium production.

These outcomes can be seen as reversing the general properties of the equilibrium analysis done in the model explored in this paper. It turns out that the inability of a low cost firm to commit to production at date 1 only, precludes the possibility that it acts as a leader.
4.5 General Demand and Cost Functions

Another direction to extend the model would be the consideration of more general demand and cost functions. The analysis done in the paper uses two main features of the model. First, that the profit function is continuous and concave. Second, that the first order conditions for profit maximization of a firm are linear in the quantities chosen by the other firm. The interesting question would then be to know to what extent the results generalize to other continuous and concave profit functions.

The central result in this paper is based on payoff comparisons, involving strict inequalities. Therefore, the result must hold if the demand and the cost functions are such that the first order conditions for profit maximization are sufficiently close to linear on the other firm’s production. Nevertheless, the generalization of the result would require a different technique for the proof of the main proposition. The current one explores the linearity of the best reply function, which are driven by the primitive assumptions on demand and costs.

Extending the results to more general demand and cost functions, would however be more than a technical curiosity. It could be used to broadly support an interpretation of the model, as describing the initial stages of a duopoly, in which the firms choose the capacity to be installed, followed by an unspecified competition.

5 Concluding Remarks

The Stackelberg and the Cournot equilibria are two central concepts in the analysis of duopolies. However, we lack theories to explain when the duopolists should choose sequentially and when they should choose simultaneously. This paper considers a simple model of a duopoly in which the firms have private information about their production costs. The firms can choose when to produce. A firm that produces at date 1 takes a leadership behavior, while a firm that produces at date 2 behaves as a follower. Hence, the model provides a natural framework to study the endogenous selection of Stackelberg and Cournot behavior.

It is shown that, in the unique symmetric cost-dependent equilibrium, a firm with a low cost will act as a leader by producing at date 1, while a firm with a high cost will produce at date 2, and hence behaves as a follower. Even though cost-independent Stackelberg equilibria also exist in the model, the previous cost-dependent equilibrium is the only one that is endogenously motivated by
the model. In this equilibrium one may observe a Cournot outcome, if both firms have low costs, a Stackelberg outcome with leadership of the low cost firm, if firms have different costs, and a double leadership outcome, if both firms have high costs.

These conclusions are thus different from those of most of the existing literature, which predict an endogenous Stackelberg outcome.

The analysis of the model in the paper is extended to the continuous type case, and it is argued that if should extend to more general demand and cost functions. Exploratory analysis of some parameterizations of the model that allow for strategic complementarities and correlation on costs suggest that the results extend to these more general frameworks.

On the other hand, in an example in which the firms are allowed to produce in both periods, it is shown that there exists an equilibrium in which the main conclusions are reversed: if both firms have low costs one gets a Cournot outcome; if firms have different costs, one gets a Stackelberg outcome with leadership of the high cost firm; if both firms have low costs, one gets a double leadership outcome. This raises an interest for the analysis of model in which the firms are allowed to produce in both periods.

Finally, the modeling approach considered in this paper can also be applied many other contexts, from industrial organization to labour markets, from international economics to applications in finance. Two such examples would be duopolies with price setting firms and the Rubinstein alternating offer bargaining game.
Appendix: Proofs

Proof of Proposition 1

Suppose, without loss of generality, that firm 1 produces at date 1, independently of its cost, i.e., \( \tau^*(c_1) = 1 \) (or equivalently \( \alpha^*(c_1) = 1 \)) for \( c_1 \in [c, \bar{c}] \). Then, it is straightforward that firm 2 is better off by producing only at date 2.

On the other hand, if firm 2 produces only at date 2, firm 1 will be better off by producing only at date 1, as long as firm 2 puts a sufficiently high probability on firm 1 having a high cost if it had not produce on date 1.

This concludes the proof.

Proof of Proposition 2

First, the following Lemma is proven:

Lemma A1 Suppose that \((\tilde{\alpha}_i, \tilde{\lambda}_i, \tilde{\phi}_i, \tilde{\xi}_i)\) is firm i’s best response to the strategy \((\alpha_j, \lambda_j, \phi_j, \xi_j)\) of firm j. If, under this best response, firm i’s expected profit from producing at date 1 is not lower than that from producing at date 2, then \( E\{\phi_j(\tilde{\lambda}_i(c_i), c_j) | \tau_j(c_j) = 2\} < E\{\xi_j(c_j) | \tau_j(c_j) = 2\} \).

Proof of Lemma A1

If \((\tilde{\alpha}_i, \tilde{\lambda}_i, \tilde{\phi}_i, \tilde{\xi}_i)\) is a best response of firm i to firm j’s strategy, and firm i’s expected profit from producing at date 1 is greater or equal to its profit from producing at date 2:

\[
\beta_j E\left\{\pi_i(\tilde{\lambda}_i(c_i), \lambda_j(c_j), c_i) | \tau_j(c_j) = 1\right\} + (1 - \beta_j) E\left\{\pi_i(\tilde{\phi}_i(\lambda_j(c_j), c_i), \tilde{\lambda}_i(c_i), c_i) | \tau_j(c_j) = 2\right\} \\
\geq \beta_j E\left\{\pi_i(\tilde{\phi}_i(\lambda_j(c_j), c_i), \lambda_j(c_j), c_i) | \tau_j(c_j) = 1\right\} + (1 - \beta_j) E\left\{\pi_i(\tilde{\xi}_i(c_i), \xi_j(c_j), c_i) | \tau_j(c_j) = 2\right\} .
\]

(A1)

But, as \( \tilde{\phi}_i(q, c_i) \) maximizes the profit of firm i when firm j has produced q:

\[
E\left\{\pi_i(\tilde{\lambda}_i(c_i), \lambda_j(c_j), c_i) | \tau_j(c_j) = 1\right\} < E\left\{\pi_i(\tilde{\phi}_i(\lambda_j(c_j), c_i), \lambda_j(c_j), c_i) | \tau_j(c_j) = 1\right\} .
\]

(A2)
From (A1) and (A2), it follows that:

$$\mathbb{E} \left\{ \pi_i(\lambda_i(c_i), \phi_j(\lambda_i(c_i), c_j), c_i) \mid \tau_j(c_j) = 2 \right\} > \mathbb{E} \left\{ \pi_i(\tilde{\lambda}_i(c_i), \xi_j(c_j), c_i) \mid \tau_j(c_j) = 2 \right\}.$$  \hspace{1cm} (A3)

Now suppose that $$\mathbb{E} \left\{ \phi_j(\tilde{\lambda}_i(c_i), c_j) \mid \tau_j(c_j) = 2 \right\} \geq \mathbb{E} \left\{ \xi_j(c_j) \mid \tau_j(c_j) = 2 \right\}$$. Then,

$$\mathbb{E} \left\{ \pi_i(\tilde{\lambda}_i(c_i), \phi_j(\tilde{\lambda}_i(c_i), c_j), c_i) \mid \tau_j(c_j) = 2 \right\}$$

$$= \pi_i(\tilde{\lambda}_i(c_i), \mathbb{E} \left\{ \phi_j(\tilde{\lambda}_i(c_i), c_j) \mid \tau_j(c_j) = 2 \right\}, c_i)$$

$$\leq \mathbb{E} \left\{ \pi_i(\tilde{\lambda}_i(c_i), \xi_j(c_j), c_i) \mid \tau_j(c_j) = 2 \right\}$$

$$\leq \mathbb{E} \left\{ \pi_i(\tilde{\xi}_i(c_i), \xi_j(c_j), c_i) \mid \tau_j(c_j) = 2 \right\}$$

which contradicts (A3).

This proves the Lemma.

We now return to the proof of the Proposition. Suppose that there exists an equilibrium with cost contingent timing where the firms use strategies $$(\lambda^*_i, \phi^*_i, \xi^*_i)$$, with $$i \in \{1, 2\}$$. Let $$\beta^*_i \in (0, 1)$$ be the probability that firm $$i$$ produces at date 1, with $$i \in \{1, 2\}$$. Define the functions $$(\tilde{\lambda}_i, \tilde{\phi}_i, \tilde{\xi}_i)$$, for $$x \in [\hat{e}, \hat{c}]$$ and $$i \neq j \in \{1, 2\}$$, through:

$$a - x - 2\tilde{\lambda}_i(x) - \left( \beta^*_j \mathbb{E} \left\{ \lambda^*_j(c_j) \mid \tau^*_j(c_j) = 1 \right\} + (1 - \beta^*_j)\mathbb{E} \left\{ \phi^*_j(\lambda_i(x), c_j) \mid \tau^*_j(c_j) = 2 \right\} \right)$$

$$- (1 - \beta^*_j)\mathbb{E} \left\{ \frac{\partial \phi^*_j(\lambda_i(x), c_j)}{\partial q_i} \lambda_i(x) \right\} \tau^*_j(c_j) = 2 \right\} = 0, \hspace{1cm} (A4)$$

$$a - x - 2\tilde{\phi}_i(q_j, x) - q_j \equiv 0, \hspace{1cm} (A5)$$

$$a - x - 2\tilde{\xi}_i(x) - \mathbb{E} \left\{ \xi^*_j(c_j) \mid \tau^*_j(c_j) = 2 \right\} \equiv 0. \hspace{1cm} (A6)$$

Clearly, $$(\tilde{\lambda}_i(x), \tilde{\phi}_i(q_j, x), \tilde{\xi}_i(x)) = (\lambda^*_i(x), \phi^*_i(q_j, x), \xi^*_i(x))$$, for $$x \in [\hat{e}, \hat{c}]$$, as (A4)-(A6) correspond then to the first order conditions for firm $$i$$’s profit maximization that define $$(\lambda^*_i, \phi^*_i, \xi^*_i)$$.

Define the function $$\tilde{\Delta}_i(x)$$, for $$x \in [\hat{e}, \hat{c}]$$ and $$i \neq j \in \{1, 2\}$$:

$$\tilde{\Delta}_i(x) \equiv \beta^*_j \mathbb{E} \left\{ \pi_i(\tilde{\lambda}_i(x), \lambda^*_j(c_j), x) \mid \tau^*_j(c_j) = 1 \right\} + (1 - \beta^*_j)\mathbb{E} \left\{ \pi_i(\tilde{\lambda}_i(x), \phi^*_j(\lambda_i(x), c_j), x) \mid \tau^*_j(c_j) = 2 \right\}$$

$$- \beta^*_j \mathbb{E} \left\{ \pi_i(\tilde{\phi}_i(\lambda^*_j(c_j), x), \lambda^*_j(c_j), x) \mid \tau^*_j(c_j) = 1 \right\} - (1 - \beta^*_j)\mathbb{E} \left\{ \pi_i(\tilde{\xi}_i(x), \xi^*_j(c_j), x) \mid \tau^*_j(c_j) = 2 \right\}.$$  \hspace{1cm} (A7)
This function is continuous and differentiable. Therefore:

\[ \tilde{\Delta}_i(c) - \bar{\Delta}_i(c) = \int_{c}^{\bar{c}} \frac{d\Delta_i(x)}{dx} dx. \]

Moreover, as there is no equilibrium in which both firms produce in the same date with probability one (if both firms were producing at date 1, one firm could increase its profit by deviating to produce at date 2 and replying optimally to the realized production of the other firm; while, if both firms were producing at date 2, one firm could increase its profit by deviating to produce at date 1 and take the benefit of being a leader), \( \tilde{\Delta}_i(c)\bar{\Delta}_i(c) \leq 0 \) and not both of them can be zero. Hence, there must exist at least one \( \hat{x}_i \in [c, \bar{c}] \), such that \( \Delta_i(\hat{x}_i) = 0 \). Using (A4) to (A7) one may write:

\[
\frac{d\Delta_i(x)}{dx} \bigg|_{x=\hat{x}_i} = -\tilde{\lambda}_i(\hat{x}_i) + \beta_j^* \mathbb{E} \left\{ \tilde{\phi}_i(\lambda_j^*(c_j), \hat{x}_i) \right\} \tau_j^*(c_j) = 1 + (1 - \beta_j^*)\tilde{\xi}_i(\hat{x}_i). \tag{A8}
\]

Multiplying (A5) by \( \beta_j^* \), and taking the expected value conditional on \( \tau_j^*(c_j) = 1 \), multiplying (A6) by \( (1 - \beta_j^*) \), summing up the two expressions, and evaluating the result at \( x = \hat{x}_i \), one obtains:

\[
a - \hat{x}_i - 2 \left( \beta_j^* \mathbb{E} \left\{ \tilde{\phi}_i(\lambda_j^*(c_j), \hat{x}_i) \right\} \tau_j^*(c_j) = 1 \right) + (1 - \beta_j^*)\tilde{\xi}_i(\hat{x}_i) \nonumber \\
- \left( \beta_j^* \mathbb{E} \left\{ \lambda_j^*(c_j) \right\} \tau_j^*(c_j) = 1 \right) + (1 - \beta_j^*)\mathbb{E} \left\{ \xi_j^*(c_j) \right\} \tau_j^*(c_j) = 2 \right) = 0. \tag{A9}
\]

But, from (A9) and Lemma A1, one may conclude that:

\[
a - \hat{x}_i - 2 \left( \beta_j^* \mathbb{E} \left\{ \tilde{\phi}_i(\lambda_j^*(c_j), \hat{x}_i) \right\} \tau_j^*(c_j) = 1 \right) + (1 - \beta_j^*)\tilde{\xi}_i(\hat{x}_i) \nonumber \\
- \left( \beta_j^* \mathbb{E} \left\{ \lambda_j^*(c_j) \right\} \tau_j^*(c_j) = 1 \right) + (1 - \beta_j^*)\mathbb{E} \left\{ \phi_j^*(\tilde{\lambda}_i(\hat{x}_i), c_j) \right\} \tau_j^*(c_j) = 2 \right) \nonumber > 0. \tag{A10}
\]

Subtracting (A10) from (A4), evaluated at \( x = \hat{x}_i \), one obtains:

\[
-2\tilde{\lambda}_i(\hat{x}_i) + 2 \left( \beta_j^* \mathbb{E} \left\{ \tilde{\phi}_i(\lambda_j^*(c_j), \hat{x}_i) \right\} \tau_j^*(c_j) = 1 \right) + (1 - \beta_j^*)\tilde{\xi}_i(\hat{x}_i) \nonumber \\
-(1 - \beta_j^*)\mathbb{E} \left\{ \frac{\partial \phi_j^*(\tilde{\lambda}_i(\hat{x}_i), c_j)}{\partial q_i} \tilde{\lambda}_i(\hat{x}_i) \right\} \tau_j^*(c_j) = 2 \right) < 0. \tag{A11}
\]

\[^4\]If \( \hat{x}_i \in \{c, \bar{c}\} \), the derivative is computed only in the direction of the interior of \([c, \bar{c}]\).
As $\partial \phi^*_j(\tilde{\lambda}_i(\hat{x}_i), c_j)/\partial q_i < 0$, (A11) implies:

$$\tilde{\lambda}_i(\hat{x}_i) > \beta^*_j E \left\{ \tilde{\phi}_i(\lambda^*_j(c_j), \hat{x}_i) \bigg| \tau^*(c_j) = 1 \right\} + (1 - \beta^*_j)\tilde{\xi}_i(\hat{x}_i).$$

(A12)

Therefore, one concludes that (A8) is negative. Hence, by continuity, $\hat{x}_i$ is unique, $\tilde{\Delta}_i(c) \geq 0$, $\tilde{\Delta}_i(\bar{c}) \leq 0$, and $\tilde{\Delta}_i(\bar{c}) - \tilde{\Delta}_i(c) < 0$. A firm with cost equal to $c$ will produce at date 1 with positive probability, a firm with cost equal to $\bar{c}$ will produce at date 2 with positive probability, and at most one type of firm will randomize over the moment of production.

Moreover, note that from equations (A4) to (A6) for each firm, one may conclude that $\tilde{\lambda}_i$, $\tilde{\phi}_i$, and $\tilde{\xi}_i$ are linear functions of $x$. Hence, it is a system of linear equations, which has a unique solution, that is symmetric, i.e., $(\tilde{\lambda}_1, \tilde{\phi}_1, \tilde{\xi}_1) = (\tilde{\lambda}_2, \tilde{\phi}_2, \tilde{\xi}_2)$.

Hence, the statement of the proposition follows.
Proof of Proposition 3

Suppose that there exists an equilibrium in which both firms follow the strategy \( \sigma^* = (\alpha^*, \lambda^*, \phi^*, \xi^*) \), with \( \alpha^*(c) = 1 \) and \( \alpha^*(\bar{c}) = 0 \).

First, consider the problem of firm \( i \), with \( i \in \{1, 2\} \), at date 2, if it did not produce at date 1. If firm \( j \), with \( j \neq i \in \{1, 2\} \), has produced the quantity \( q_j \) at date 1, firm \( i \) will solve the problem:

\[
\max_{q_i} \ (a - q_i - q_j - c_i) q_i,
\]

which leads to the production of:

\[
\phi^*(q_j, c_i) \equiv \frac{a - c_i - q_j}{2}.
\] (A13)

Note that the expression of the optimal response is independent of the firm’s identity.

If firm \( j \), with \( j \neq i \in \{1, 2\} \), has not produced at date 1, firm \( i \) will infer that firm \( j \) has cost equal to \( \bar{c} \), and so firm \( i \) will solve the problem:

\[
\max_{q_i} \ (a - q_i - \xi^*(\bar{c}) - c_i) q_i,
\]

which leads to the production of:

\[
\xi^*(c_i) \equiv \phi^*(\xi^*(\bar{c}), c_i).
\]

Again, the expression for the optimal response is independent of the firm’s identity, so that:

\[
\xi^*(\bar{c}) \equiv \frac{a - \bar{c} - \xi^*(\bar{c})}{2} \Leftrightarrow \xi^*(\bar{c}) \equiv \frac{a - \bar{c}}{3},
\] (A14)

and

\[
\xi^*(c) \equiv \frac{a - c - \xi^*(\bar{c})}{2} \Leftrightarrow \xi^*(c) \equiv \frac{2(a - \bar{c}) + 3(\bar{c} - c)}{6}.
\] (A15)

Now, consider the problem of firm \( i \), with \( i \in \{1, 2\} \), if it produces at date 1. Given the assumed strategy, firm \( j \), \( j \neq i \in \{1, 2\} \), will produce \( \lambda^*(c) \) at date 1, if \( c_j = c \), and it will produce \( \phi^*(q_i, \bar{c}) \)
at date 2, when \( c_j = \bar{c} \) and firm \( i \) has produced \( q_i \) at date 1. Therefore, firm \( i \) solves the problem:

\[
\max_{q_i} \gamma (a - q_i - \lambda^*(c) - c_i) q_i + (1 - \gamma) (a - q_i - \phi^*(q_i, \bar{c}) - c_i) q_i.
\]

This leads to the first-order conditions:

\[
a - 2\lambda^*(c_i) - c_i - (\gamma \lambda^*(c) + (1 - \gamma) \phi^*(\lambda^*(c_i), \bar{c})) + \frac{1}{2} (1 - \gamma) \lambda^*(c_i) \equiv 0, \quad i \in \{1, 2\} \text{ and } c_i \in \{c, \bar{c}\}.
\]

This system of linear identities in \( \lambda^*(\cdot) \) has a unique solution. Solving it for each possible value of \( c_i \):

\[
a - 2\lambda^*(c) - \bar{c} - (\gamma \lambda^*(c) + (1 - \gamma) \phi^*(\lambda^*(c), \bar{c})) + \frac{1}{2} (1 - \gamma) \lambda^*(c) \equiv 0,
\]

\[
\iff \lambda^*(c) \equiv \frac{(1 + \gamma)(a - \bar{c}) + 2(\bar{c} - c)}{2(1 + 2\gamma)}, \quad (A16)
\]

and:

\[
a - 2\lambda^*(\bar{c}) - \bar{c} - (\gamma \lambda^*(c) + (1 - \gamma) \phi^*(\lambda^*(\bar{c}), \bar{c})) + \frac{1}{2} (1 - \gamma) \lambda^*(\bar{c}) \equiv 0,
\]

\[
\iff \lambda^*(\bar{c}) \equiv \frac{(1 + \gamma)^2(a - \bar{c}) - 2(\bar{c} - \bar{c})}{2(1 + \gamma)(1 + 2\gamma)}. \quad (A17)
\]

The optimal moment of production is determined by comparing the associated expected profits:

1. When firm \( i \), with \( i \in \{1, 2\} \), has cost equal to \( c \):

   (a) If firm \( i \) produces at date 1, its expected profit will be:

   \[
   \Pi^1_i(c, \sigma^*) = \frac{(1 + \gamma)(1 + \gamma)(a - \bar{c}) + 2(\bar{c} - \bar{c})^2}{8(1 + 2\gamma)^2}.
   \]

   (b) If firm \( i \) produces at date 2, its expected profit will be:

   \[
   \Pi^2_i(c, \sigma^*) = \gamma \left(\frac{1 + 3\gamma}{4 + 8\gamma}(a - \bar{c}) + 4\gamma(\bar{c} - \bar{c})\right)^2 + (1 - \gamma) \left(\frac{2(a - \bar{c}) + 3(\bar{c} - \bar{c})}{6}\right)^2.
   \]
Comparing the two possible payoffs of firm $i$, one obtains:

\[
\begin{align*}
\frac{(1 + \gamma)(1 + \gamma)(a - \bar{c}) + 2(\bar{c} - c))^2}{8(1 + 2\gamma)^2} \\
\quad - \gamma \left( \frac{(1 + 3\gamma)(a - \bar{c}) + 4\gamma(\bar{c} - c)}{4 + 8\gamma} \right)^2 - (1 - \gamma) \left( \frac{2(a - \bar{c}) + 3(\bar{c} - c)}{6} \right)^2 \\
= \frac{(1 - \gamma)^2(2 + \gamma)(a - \bar{c})^2 + 24(1 - \gamma^3)(a - \bar{c})(\bar{c} - c) + 36(1 - \gamma)(\bar{c} - c)^2}{144(1 + 2\gamma)^2},
\end{align*}
\]

which, given the restrictions on the parameters, is positive. So, firm $i$’s expected profit from producing at date 1 is greater than that from producing at date 2. Hence, $\tau^*(\bar{c}) = 1$ (or $\alpha^*(c) = 1$).

2. When firm $i$ has cost equal to $\bar{c}$:

   (a) If firm $i$ produces at date 1, its expected profit will be:

   \[
   \Pi^1_i(\bar{c}, \sigma^*) = \frac{((1 + \gamma)^2(a - \bar{c}) - 2\gamma(\bar{c} - c))^2}{8(1 + \gamma)(1 + 2\gamma)^2}.
   \]

   (b) If firm $i$ produces at date 2, its expected profit will be:

   \[
   \Pi^2_i(\bar{c}, \sigma^*) = \gamma \left( \frac{(1 + 3\gamma)(a - \bar{c}) - 2(\bar{c} - c)}{4(1 + 2\gamma)} \right)^2 + (1 - \gamma) \left( \frac{a - \bar{c}}{3} \right)^2.
   \]

Comparing the two expected profits of firm $i$, one concludes that firm $i$ will prefer to produce at date 2, so that $\tau^*(\bar{c}) = 2$ (or $\alpha^*(\bar{c}) = 0$), if:

\[
\frac{(1 - \gamma)(1 + \gamma)(2 + \gamma)(\gamma - 1)(a - \bar{c})^2 + 36\gamma(1 + \gamma)(a - \bar{c})(\bar{c} - c) + 36\gamma(\bar{c} - c)^2}{144(1 + \gamma)(1 + 2\gamma)^2} \geq 0,
\]

which is equivalent to:

\[
\frac{\bar{c} - c}{a - c} \geq \frac{(1 + \gamma)(2 + 17\gamma - \gamma^2) - 6\sqrt{2}\gamma(1 + \gamma)(1 + 2\gamma)}{2 + \gamma + 34\gamma^2 - \gamma^3}.
\]

Given the restrictions on the parameters, the right hand side of this last condition is in $[0, 1]$, which concludes the proof.
Proof of Proposition 4

Suppose that there exists an equilibrium in which both firms follow the strategy \( \sigma^* = (\alpha^*, \lambda^*, \phi^*, \xi^*) \), with \( \alpha^*(c) = 1 \) and \( \alpha^*(\bar{c}) = \hat{\alpha} \), with \( 0 < \hat{\alpha} < 1 \).

The problem of firm \( i \), with \( i \in \{1, 2\} \), at date 2 if it did not produce at date 1 is exactly the same of Proposition 3. Hence:

\[
\phi^*(q_j, c_i) \equiv \frac{a - c_i - q_j}{2} \quad (A18)
\]

\[
\xi^*(\bar{c}) \equiv \frac{a - \bar{c}}{3} \quad (A19)
\]

\[
\xi^*(c) \equiv \frac{2(a - \bar{c}) + 3(\bar{c} - c)}{6} \quad (A20)
\]

Now, consider the problem of firm \( i \), with \( i \in \{1, 2\} \), if it produces at date 1. Given the assumed strategy, firm \( j \), with \( j \neq i \in \{1, 2\} \), will produce \( \lambda^*(c) \) at date 1, if \( c_j = c \); if \( c_j = \bar{c} \), it will produce \( \lambda^*(\bar{c}) \) at date 1, with probability \( \hat{\alpha} \), and \( \phi^*(q_i, \bar{c}) \) at date 2, with probability \( 1 - \hat{\alpha} \), after firm \( i \) has produced \( q_i \) at date 1. Therefore, firm \( i \) solves the problem:

\[
\max_{q_i} \gamma (a - q_i - \lambda^*(c) - c_i) q_i + (1 - \gamma)\hat{\alpha} (a - q_i - \lambda^*(\bar{c}) - c_i) q_i + (1 - \gamma)(1 - \hat{\alpha}) (a - q_i - \phi^*(q_i, \bar{c}) - c_i) q_i .
\]

This leads to the first-order conditions:

\[
a - 2\lambda^*(c_i) - c_i - (\gamma \lambda^*(c) + (1 - \gamma)\hat{\alpha} \lambda^*(\bar{c}) + (1 - \gamma)(1 - \hat{\alpha})\phi^*(\lambda^*(c_i), \bar{c}))
+ \frac{1}{2}(1 - \gamma)(1 - \hat{\alpha})\lambda^*(c_i) \equiv 0, i \neq j \in \{1, 2\} \text{ and } c_i \in \{c, \bar{c}\}.
\]

This system of linear identities in \( \lambda^*(c_i) \), with \( i \in \{1, 2\} \), has a unique solution given by:

\[
\lambda^*(c) \equiv \frac{(1 + \gamma + (1 - \gamma)\hat{\alpha})^2(a - \bar{c}) + 2(1 + \gamma + 2(1 - \gamma)\hat{\alpha})(\bar{c} - c)}{2(1 + \gamma + (1 - \gamma)\hat{\alpha})(1 + 2\gamma + 2(1 - \gamma)\hat{\alpha})} \quad (A21)
\]

\[
\lambda^*(\bar{c}) \equiv \frac{(1 + \gamma + (1 - \gamma)\hat{\alpha})^2(a - \bar{c}) - 2\gamma(\bar{c} - c)}{2(1 + \gamma + (1 - \gamma)\hat{\alpha})(1 + 2\gamma + 2(1 - \gamma)\hat{\alpha})} \quad (A22)
\]
As a result, if firm $i$ has cost equal to $\bar{c}$ and it produces at date 1, its expected profit will be:

$$\Pi_1^i(\bar{c}, \sigma^*) = \frac{((1 + \gamma + (1 - \gamma)\hat{\alpha})(a - \bar{c}) - 2\gamma(\bar{c} - c))^2}{8(1 + \gamma + (1 - \gamma)\hat{\alpha})(1 + 2\gamma + 2(1 - \gamma)\hat{\alpha})^2}, \quad (A23)$$

while, if it produces at date 2, its expected profit will be:

$$\Pi_2^i(\bar{c}, \sigma^*) = \gamma \left( \frac{(1 + \gamma + (1 - \gamma)\hat{\alpha})(1 + 3\gamma + 3(1 - \gamma)\hat{\alpha})(a - \bar{c}) - 2(1 + \gamma + 2(1 - \gamma)\hat{\alpha})(\bar{c} - c))}{4(1 + \gamma + (1 - \gamma)\hat{\alpha})(1 + 2\gamma + 2(1 - \gamma)\hat{\alpha})} \right)^2$$

$$+ (1 - \gamma)\hat{\alpha} \left( \frac{(1 + \gamma + (1 - \gamma)\hat{\alpha})(1 + 3\gamma + 3(1 - \gamma)\hat{\alpha})(a - \bar{c}) + 2\gamma(\bar{c} - c))}{4(1 + \gamma + (1 - \gamma)\hat{\alpha})(1 + 2\gamma + 2(1 - \gamma)\hat{\alpha})} \right)^2$$

$$+ (1 - \gamma)(1 - \hat{\alpha}) \left( \frac{a - \bar{c}}{3} \right)^2. \quad (A24)$$

The probability $\hat{\alpha}$ will be such that a firm with cost equal to $\bar{c}$ will be indifferent between producing at date 1 or producing at date 2. It is not possible to obtain a closed form solution for $\hat{\alpha}$, but numerical methods allow the conclusion that there is a unique value of $\hat{\alpha}$ in $[0, 1]$ equalizing the two profits.

To conclude the proof, it should be checked that, under these conditions, if firm $i$ has cost equal to $\bar{c}$, it will prefer to produce at date 1. Calculations lead to:

$$\left( \Pi_1^i(\bar{c}, \sigma^*) - \Pi_2^i(\bar{c}, \sigma^*) \right) - \left( \Pi_1^i(\bar{c}, \sigma^*) - \Pi_2^i(\bar{c}, \sigma^*) \right)$$

$$= \frac{(1 - \hat{\alpha})(1 - \gamma)((1 + \gamma + (1 - \gamma)\hat{\alpha})(2 + \gamma + (1 - \gamma)\hat{\alpha})(a - \bar{c}) + 3(1 + 2(1 - \gamma)\hat{\alpha})(\bar{c} - c))(\bar{c} - c))}{12(1 + \gamma + (1 - \gamma)\hat{\alpha})(1 + 2\gamma + 2(1 - \gamma)\hat{\alpha})}$$

$$\geq 0.$$ 

But, as $\hat{\alpha}$ is such that $\Pi_1^i(\bar{c}, \sigma^*) - \Pi_2^i(\bar{c}, \sigma^*) = 0$, the condition implies that $\Pi_1^i(\bar{c}, \sigma^*) - \Pi_2^i(\bar{c}, \sigma^*) \geq 0$.

Hence, if firm $i$ has cost equal to $\bar{c}$, it will prefer to produce at date 1.

This concludes the proof.
Proof of Corollary 4.1

The asymmetry of information in the model vanishes as long as at least one of the following conditions is satisfied: $\gamma \to 0$, $\gamma \to 1$, or $\bar{c} - \bar{\xi} \to 0$.

Suppose that $\gamma \to 0$ and that there exists $\epsilon > 0$ such that $\bar{c} - \bar{\xi} > \epsilon$. Then, there will be a $\delta > 0$ such that, for $\gamma < \delta$, the condition in Proposition 4 will be satisfied. Using expressions (A23) and (A24) one may conclude that, under these conditions, $\hat{\alpha} \to 1$. Then, taking almost sure convergence, one needs only to consider the limit of the quantity produced by a high cost firm in date 1; considering (A22), one concludes that, under these conditions, $\lambda^* (\bar{c}) \to (a - \bar{c}) / 3$. Hence, in the limit, both firms will produce the quantity $(a - \bar{c}) / 3$ at date 1, where $\bar{c}$ is the (limit) common cost of the two firms.

Suppose now that $\gamma \to 1$ and that there exists $\epsilon > 0$ such that $\bar{c} - \bar{\xi} > \epsilon$. Then, there will be a $\delta > 0$ such that, for $\gamma > 1 - \delta$, the condition in Proposition 3 will be satisfied. Then, taking almost sure convergence, one needs only to consider the limit of the quantity produced by a low cost firm in date 1; considering expression (A16), one concludes that, under these conditions, $\lambda^* (\bar{c}) \to (a - \bar{c}) / 3$. Hence, in the limit, both firms will produce the quantity $(a - \bar{c}) / 3$ at date 1, where $\bar{c}$ is the (limit) common cost of the two firms.

Consider now the case of $\bar{c} - \bar{\xi} \to 0$ with the existence of $\epsilon > 0$ such that $\gamma < 1 - \epsilon$ (this is trivially satisfied if $\gamma \to 0$). Then, there will be a $\delta > 0$ such that, for $\bar{c} - \bar{\xi} < \delta$, the condition of Proposition 4 will be satisfied. Using expressions (A23) and (A24) one may conclude that, under these conditions, $\hat{\alpha} \to 1$. Then, as $\gamma$ is not necessarily converging to 0, taking almost sure convergence, one needs to consider the limit of the quantity produced in date 1 by both a low cost and a high cost firm; then, considering (A21) and (A22), one concludes that, under these conditions, $\lambda^* (\bar{c}) \to (a - \bar{c}) / 3$ and $\lambda^* (\bar{\xi}) \to (a - \bar{\xi}) / 3$, where $\bar{c}$ is the (limit) common cost of the two firms. Hence, both firms will produce the quantity $(a - \bar{c}) / 3$ at date 1.

Finally, consider the case of $\bar{c} - \bar{\xi} \to 0$ and $\gamma \to 1$. It is possible that, as the parameters go to their limits, either the condition on Proposition 3 or the condition on Proposition 4 be satisfied. If we take the subset of these parameter values that satisfy the condition on Proposition 3, using expressions (A16) and (A17), one concludes that, under these conditions, $\lambda^* (\bar{c}) \to (a - \bar{c}) / 3$ and $\lambda^* (\bar{\xi}) \to (a - \bar{\xi}) / 3$, where $\bar{c}$ is the (limit) common cost of the two firms. For the subset of these parameter values that satisfy the condition on Proposition 4, using expressions (A21) and (A22),
one also concludes that, under these conditions, \( \lambda^*(c) \rightarrow (a - c)/3 \) and \( \lambda^*(\bar{c}) \rightarrow (a - c)/3 \), where \( c \) is the (limit) common cost of the two firms. Hence, both firms will produce the quantity \( (a - c)/3 \) at date 1.

This concludes the proof.
Proof of Proposition 5

The proof reproduces the steps of the proof of Proposition 2. From the fact that (A8) is negative one concludes that \( \hat{x}_i \). Moreover, the equilibrium strategy is the same for both firms. Therefore, let \( c^* = \hat{x}_1 = \hat{x}_2 \), and the proposition follows.
Proof of Proposition 6

The proof reproduces the proof of Proposition 2, and some steps are omitted. First, the following Lemma is proven:

**Lemma A2** Consider the model with differentiated goods. Suppose that \((\hat{\alpha}_i, \hat{\lambda}_i, \hat{\phi}_i, \hat{\xi}_i)\) is firm \(i\)'s best response to the strategy \((\alpha_j, \lambda_j, \phi_j, \xi_{ji})\) of firm \(j\). If, under this best response, firm \(i\)'s expected profit from producing at date 1 is not lower than that from producing at date 2, then

\[
\theta E \left\{ \phi_j(\hat{\lambda}_i(c_i), c_j) \mid \tau_j = 2 \right\} > \theta E \left\{ \xi_j(c_j) \mid \tau_j = 2 \right\}.
\]

Proof of Lemma A2

If \((\hat{\alpha}_i, \hat{\lambda}_i, \hat{\phi}_i, \hat{\xi}_i)\) is a best response of firm \(i\) to firm \(j\)'s strategy, and firm \(i\)'s expected profit from producing at date 1 is greater or equal to its profit from producing at date 2:

\[
\beta_j \mathbb{E} \left\{ \pi_i(\hat{\lambda}_i(c_i), \lambda_j(c_j), c_i) \mid \tau_j(c_j) = 1 \right\} + (1 - \beta_j) \mathbb{E} \left\{ \pi_i(\hat{\lambda}_i(c_i), \phi_j(\hat{\lambda}_i(c_i), c_j), c_i) \mid \tau_j(c_j) = 2 \right\} \\
\geq \beta_j \mathbb{E} \left\{ \pi_i(\hat{\phi}_i(\lambda_j(c_j), c_i), \lambda_j(c_j), c_i) \mid \tau_j(c_j) = 1 \right\} + (1 - \beta_j) \mathbb{E} \left\{ \pi_i(\hat{\xi}_i(c_i), \xi_j(c_j), c_i) \mid \tau_j(c_j) = 2 \right\}.
\]

(A25)

But, as \(\hat{\phi}_i(q, c_i)\) maximizes the profit of firm \(i\) when firm \(j\) has produced \(q\):

\[
\mathbb{E} \left\{ \pi_i(\hat{\lambda}_i(c_i), \lambda_j(c_j), c_i) \mid \tau_j(c_j) = 1 \right\} < \mathbb{E} \left\{ \pi_i(\hat{\phi}_i(\lambda_j(c_j), c_i), \lambda_j(c_j), c_i) \mid \tau_j(c_j) = 1 \right\}.
\]

(A26)

From (A25) and (A26), it follows that:

\[
\mathbb{E} \left\{ \pi_i(\hat{\lambda}_i(c_i), \phi_j(\hat{\lambda}_i(c_i), c_j), c_i) \mid \tau_j(c_j) = 2 \right\} > \mathbb{E} \left\{ \pi_i(\hat{\xi}_i(c_i), \xi_j(c_j), c_i) \mid \tau_j(c_j) = 2 \right\}.
\]

(A27)

But, given the linearity of firm 1’s profit function on firm 2’s output, (A27) is possible only if:

\[
\theta \mathbb{E} \left\{ \phi_j(\hat{\lambda}_i(c_i), c_j) \mid \tau_j(c_j) = 2 \right\} > \theta \mathbb{E} \left\{ \xi_j(c_j) \mid \tau_j(c_j) = 2 \right\}.
\]

(A28)

This proves the Lemma.

Suppose that there exists a time-dependent equilibrium where the firms use strategies \((\alpha_i^*, \lambda_i^*, \phi_i^*, \xi_i^*)\), with \(i \in \{1, 2\}\). Let \(\beta_i^* \in (0, 1)\) be the probability that firm \(i\) produces at date 1, with \(i \in \{1, 2\}\).
Define the functions \((\tilde{\lambda}_i, \tilde{\phi}_i, \tilde{\xi}_i)\), for \(x \in [\tilde{c}, \bar{c}]\) and \(i \in \{1, 2\}\), through:

\[
a - x - 2\tilde{\lambda}_i(x) + \theta \left( \beta_j^* E \left\{ \lambda_j^*(c_j) \big| \tau_j^*(c_j) = 1 \right\} + (1 - \beta_j^*) E \left\{ \phi_j^*(\tilde{\lambda}_i(x), c_j) \big| \tau_j^*(c_j) = 2 \right\} \right) \\
+ (1 - \beta_j^*) \theta E \left\{ \frac{\partial \phi_j^*(\tilde{\lambda}_i(x), c_j)}{\partial q_i} \tilde{\lambda}_i(x) \big| \tau_j^*(c_j) = 2 \right\} \equiv 0, \tag{A29}
\]

\[
a - x - 2\tilde{\phi}_i(q_j, x) + \theta q_j \equiv 0, \tag{A30}
\]

\[
a - x - 2\tilde{\xi}_i(x) + \theta E \left\{ \xi_j^*(c_j) \big| \tau_j^*(c_j) = 2 \right\} \equiv 0. \tag{A31}
\]

Clearly, \((\tilde{\lambda}_i(x), \tilde{\phi}_i(q_j, x), \tilde{\xi}_i(x)) = (\lambda_i^*(x), \phi_i^*(q_j, x), \xi_i^*(x))\), for \(x \in [\tilde{c}, \bar{c}]\), as \((A29)\)-(A31) correspond then to the first order conditions for firm \(i\)’s profit maximization that define \((\lambda_i^*, \phi_i^*, \xi_i^*)\).

For \(i \in \{1, 2\}\), define the function \(\tilde{\Delta}_i(x)\), for \(x \in [\tilde{c}, \bar{c}]\), and \(\hat{x}_i\) as in the proof of Proposition 2. Hence:

\[
\left. \frac{d\tilde{\Delta}_i(x)}{dx} \right|_{x=\hat{x}_i} = -\tilde{\lambda}_i(\hat{x}_i) + \beta_j^* E \left\{ \tilde{\phi}_i(\lambda_j^*(c_j), \hat{x}_i) \big| \tau_j^*(c_j) = 1 \right\} + (1 - \beta_j^*) \tilde{\xi}_i(\hat{x}_i). \tag{A32}
\]

Multiplying condition \((A30)\) by \(\beta_j^*\) and taking the expected value conditional on \(\tau_j^*(c_j) = 1\), condition \((A31)\) by \((1 - \beta_j^*)\), summing up the two, and evaluating it at \(x = \hat{x}_i\), one obtains:

\[
a - \hat{x}_i - 2 \left( \beta_j^* E \left\{ \tilde{\phi}_i(\lambda_j^*(c_j), \hat{x}_i) \big| \tau_j^*(c_j) = 1 \right\} + (1 - \beta_j^*) \tilde{\xi}_i(\hat{x}_i) \right) \\
+ \theta \left( \beta_j^* E \left\{ \lambda_j^*(c_j) \big| \tau_j^*(c_j) = 1 \right\} + (1 - \beta_j^*) E \left\{ \phi_j^*(\tilde{\lambda}_i(x), c_j) \big| \tau_j^*(c_j) = 2 \right\} \right) = 0. \tag{A33}
\]

But, from \((A33)\) and Lemma A2, one may conclude that:

\[
a - \hat{x}_i - 2 \left( \beta_j^* E \left\{ \tilde{\phi}_i(\lambda_j^*(c_j), \hat{x}_i) \big| \tau_j^*(c_j) = 1 \right\} + (1 - \beta_j^*) \tilde{\xi}_i(\hat{x}_i) \right) \\
+ \theta \left( \beta_j^* E \left\{ \lambda_j^*(c_j) \big| \tau_j^*(c_j) = 1 \right\} + (1 - \beta_j^*) E \left\{ \phi_j^*(\tilde{\lambda}_i(x), c_j) \big| \tau_j^*(c_j) = 2 \right\} \right) > 0. \tag{A34}
\]

So, from \((A29)\), \((A34)\), and the fact that \(\theta \partial \tilde{\phi}_j^*(\tilde{\lambda}_i(\hat{x}_i), c_j) / \partial q_i > 0:\)

\[
\tilde{\lambda}_i(\hat{x}_i) > \beta_j^* E \left\{ \tilde{\phi}_i(\lambda_j^*(c_j), \hat{x}_i) \big| \tau_j^*(c_j) = 1 \right\} + (1 - \beta_j^*) \tilde{\xi}_i(\hat{x}_i). \tag{A35}
\]

Therefore, one concludes that \((A32)\) is negative. Hence, by continuity, \(\hat{x}_i\) is unique, \(\tilde{\Delta}_i(\bar{c}) \geq 0\), \(\tilde{\Delta}_i(\tilde{c}) \leq 0\), and \(\tilde{\Delta}_i(\bar{c}) - \tilde{\Delta}_i(\tilde{c}) < 0\). A firm with cost equal to \(\tilde{c}\) will produce at date 1 with positive
probability, a firm with cost equal to $\bar{c}$ will produce at date 2 with positive probability, and at most one type of firm will randomize over the moment of production.

Moreover, note that from equations (A29) to (A31) for each firm, one may conclude that $\tilde{\lambda}_i$, $\tilde{\phi}_i$, and $\tilde{\xi}_i$ are linear functions of $x$. Hence, it is a system of linear equations, which has a unique solution, that is symmetric, i.e., $(\tilde{\lambda}_1, \tilde{\phi}_1, \tilde{\xi}_1) = (\tilde{\lambda}_2, \tilde{\phi}_2, \tilde{\xi}_2)$.

Hence, the statement of the proposition follows.
Proof of the Proposition 7

Suppose that there exists an equilibrium in which both firms follow the strategy $\sigma^* = (\alpha^*, \lambda^*, \phi^*, \xi^*)$, with $\alpha^*(c) = 1$ and $\alpha^*(\bar{c}) = 0$.

First, consider the problem of firm $i$, with $i \in \{1, 2\}$, at date 2, if it did not produce at date 1.

If firm $j$, with $j \neq i \in \{1, 2\}$, has produced the quantity $q_j$ at date 1, firm $i$ will solve the problem:

$$\max_{q_i} (a - q_i + \theta q_j - c_i) q_i,$$

which leads to the production of:

$$\phi^*(q_j, c_i) \equiv \frac{a - c_i + \theta q_j}{2}. \quad (A36)$$

Note that the expression of the optimal response is independent of the firm’s identity.

If firm $j$, with $j \neq i \in \{1, 2\}$, has not produced at date 1, firm $i$ will infer that firm $j$ has cost equal to $\bar{c}$, and so firm $i$ will solve the problem:

$$\max_{q_i} (a - q_i + \theta \xi^*(\bar{c}) - c_i) q_i,$$

which leads to the production of:

$$\xi^*(c_i) \equiv \phi^*(\xi^*(\bar{c}), c_i).$$

Again, the expression for the optimal response is independent of the firm’s identity, so that:

$$\xi^*(\bar{c}) \equiv \frac{a - \bar{c} + \theta \xi^*(\bar{c})}{2} \Rightarrow \xi^*(\bar{c}) \equiv \frac{a - \bar{c}}{2 - \theta}, \quad (A37)$$

and

$$\xi^*(c) \equiv \frac{a - c + \theta \xi^*(\bar{c})}{2} \Rightarrow \xi^*(c) \equiv \frac{2(a - \bar{c}) + (2 - \theta)(\bar{c} - c)}{2(2 - \theta)}. \quad (A38)$$

Now, consider the problem of firm $i$, with $i \in \{1, 2\}$, if it produces at date 1. Given the assumed strategy, firm $j$, $j \neq i \in \{1, 2\}$, will produce $\lambda^*(c)$ at date 1, if $c_j = c$, and it will produce $\phi^*(q_i, \bar{c})$
at date 2, when \( c_j = \bar{c} \) and firm \( i \) has produced \( q_i \) at date 1. Therefore, firm \( i \) solves the problem:

\[
\max_{q_i} \gamma (a - q_i + \theta \lambda^* (c) - c_i) q_i + (1 - \gamma) (a - q_i + \theta \phi^* (q_i, \bar{c}) - c_i) q_i .
\]

This leads to the first-order conditions:

\[
a - 2\lambda^* (c_i) - c_i + \theta (\gamma \lambda^* (\bar{c}) + (1 - \gamma) \phi^* (\lambda^* (c_i), \bar{c})) - \frac{\theta}{2} (1 - \gamma) \lambda^* (c_i) \equiv 0, \quad i \in \{1, 2\} \text{ and } c_i \in \{c, \bar{c}\}.
\]

This system of linear identities in \( \lambda^* (\cdot) \) has a unique solution. Solving it for each possible value of \( c_i \):

\[
a - 2\lambda^* (c) - c + \theta (\gamma \lambda^* (c) + (1 - \gamma) \phi^* (\lambda^* (c), \bar{c})) - \frac{\theta}{2} (1 - \gamma) \lambda^* (c) \equiv 0,
\]

\[
\Leftrightarrow \lambda^* (c) \equiv \frac{(2 + (1 - \gamma) \theta - \bar{c})}{(2(2 - \gamma \theta - (1 - \gamma) \theta^2))}, \quad (A39)
\]

and:

\[
a - 2\lambda^* (\bar{c}) - \bar{c} + \theta (\gamma \lambda^* (\bar{c}) + (1 - \gamma) \phi^* (\lambda^* (\bar{c}), \bar{c})) - \frac{\theta}{2} (1 - \gamma) \lambda^* (\bar{c}) \equiv 0,
\]

\[
\Leftrightarrow \lambda^* (\bar{c}) \equiv \frac{(2 + (1 - \gamma) \theta)(a - \bar{c}) + 2\theta (\bar{c} - c)}{2(2 - \gamma \theta - (1 - \gamma) \theta^2)} . \quad (A40)
\]

The optimal moment of production is determined by comparing the associated expected profits:

1. When firm \( i \), with \( i \in \{1, 2\} \), has cost equal to \( c \):

   (a) If firm \( i \) produces at date 1, its expected profit will be:

   \[
   \Pi^1_i(c, \sigma^*) = \frac{(2 - (1 - \gamma) \theta)(a - \bar{c}) + 2(\bar{c} - c)}{8(2 - \gamma \theta - (1 - \gamma) \theta^2)} .
   \]

   (b) If firm \( i \) produces at date 2, its expected profit will be:

   \[
   \Pi^2_i(c, \sigma^*) = \gamma \left( \frac{(4 + (1 - \gamma)(2 - \theta) \theta)(a - \bar{c}) + 2(2 + (1 - \gamma)(1 - \theta) \theta)(\bar{c} - c)}{4(2 - \gamma \theta - (1 - \gamma) \theta^2)} \right)^2 
   \]

   \[
   + (1 - \gamma) \left( \frac{2(a - \bar{c}) + (2 - \theta)(\bar{c} - c)}{2(2 - \theta)} \right)^2 .
   \]
Comparing the two possible payoffs of firm $i$, one obtains:

\[
\frac{(2 - (1 - \gamma)\theta^2) ((2 + (1 - \gamma)\theta)(a - \bar{c}) + 2(\bar{c} - c))^2}{8(2 - \gamma\theta - (1 - \gamma)\theta^2)^2} \\
- \gamma \left( \frac{(4 + (1 - \gamma)(2 - \theta)(a - \bar{c}) + 2(2 + (1 - \gamma)(1 - \theta)(\bar{c} - c)))}{4(2 - \gamma\theta - (1 - \gamma)\theta^2)} \right)^2 \\
- (1 - \gamma) \left( \frac{2(a - \bar{c}) + (2 - \theta)(\bar{c} - c)}{2(2 - \theta)} \right)^2 \\
= \frac{(1 - \gamma)\theta^2}{16(2 - \theta)^2(2 - \gamma\theta - (1 - \gamma)\theta^2)^2} \times \left( (4 - (2 - \gamma)\theta^2)(1 - \gamma)\theta^2(a - \bar{c})^2 \\
+ 4(2 - \theta)(4 - 2\gamma\theta - (2 + \gamma)(1 - \gamma)\theta^2 + \gamma(1 - \gamma)\theta^3)(a - \bar{c})(\bar{c} - c) \\
+ 4(2 - \theta)^2(2 - \gamma - (1 - \gamma)\theta^2)(\bar{c} - c)^2 \right)
\]

which, given the restrictions on the parameters, is positive. So, firm $i$’s expected profit from producing at date 1 is greater than that from producing at date 2. Hence, $\tau^*(\bar{c}) = 1$ (or $\alpha^*(\bar{c}) = 1$).

2. When firm $i$ has cost equal to $\bar{c}$:

(a) If firm $i$ produces at date 1, its expected profit will be:

\[
\Pi_i^1(\bar{c}, \sigma^*) = \frac{((2 + (1 - \gamma)\theta)(2 - (1 - \gamma)\theta^2)(a - \bar{c}) + 2\gamma\theta(\bar{c} - c))^2}{8(2 - (1 - \gamma)\theta^2)(2 - \gamma\theta - (1 - \gamma)\theta^2)^2}.
\]

(b) If firm $i$ produces at date 2, its expected profit will be:

\[
\Pi_i^2(\bar{c}, \sigma^*) = \gamma \left( \frac{(2 + (1 - \gamma)\theta)(2 - (1 - \gamma)\theta^2)(a - \bar{c}) + 2\theta(\bar{c} - c)}{4(2 - \gamma\theta - (1 - \gamma)\theta^2)} \right)^2 + (1 - \gamma) \left( \frac{a - \bar{c}}{2 - \theta} \right)^2.
\]

Comparing the two expected profits of firm $i$, one concludes that firm $i$ will prefer to produce at date 2, so that $\tau^*(\bar{c}) = 2$ (or $\alpha^*(\bar{c}) = 0$), if:

\[
\frac{\theta^2(1 - \gamma)}{16(2 - (1 - \gamma)\theta^2)(2 - \gamma\theta - (1 - \gamma)\theta^2)^2(2 - \theta)^2} \times \left( (\gamma - 1)\theta^2(8 - 2(4 - 3\gamma)\theta^2 + (1 - \gamma)(2 - \gamma)\theta^4)(a - \bar{c})^2 \\
- 4\gamma(2 - \theta)^2(2 - (1 - \gamma)\theta^2)(a - \bar{c})(\bar{c} - c) \right)
\]
\[ +4\gamma(2 - \theta)^2(2 - \theta^2)(\bar{c} - \bar{c})^2 \geq 0 , \]

which is equivalent to:

\[
\frac{\bar{c} - c}{a - c} \geq \frac{\theta((2 - (1 - \gamma)\theta^2)(8\gamma - 4(1 + \gamma)\theta + 2\gamma\theta^2 + (1 - \gamma)(2 - \gamma)\theta^4) + \frac{a}{\theta^2}2\sqrt{2\gamma(2 - (1 - \gamma)\theta^2)(2 - \theta)(2 - \gamma\theta - (1 - \gamma)\theta^2)})}{(32\gamma - 8(1 + 4\gamma)\theta^2 + 8\gamma(1 + 2\gamma)\theta^3 + 2(1 + \gamma)(4 - 5\gamma)\theta^4 - 4\gamma(1 - \gamma)\theta^5 - (1 - \gamma)^2(2 - \gamma)\theta^6)} .
\]

Given the restrictions on the parameters, the right hand side of this last condition is in \([0, 1]\), which concludes the proof.
Proof of Proposition 8

Suppose that there exists an equilibrium in which both firms follow the strategy \((\alpha^*, \lambda^*, \phi^*, \xi^*)\), with \(\alpha^*(c) = 1\) and \(\alpha^*(\bar{c}) = \bar{\alpha}\), with \(0 < \bar{\alpha} < 1\).

The problem of firm \(i\), with \(i \in \{1, 2\}\), at date 2 if it did not produce at date 1 is exactly the same of Proposition 7. Hence:

\[
\phi^*(q_j, c_i) = \frac{a - c_i + \theta q_j}{2},
\]
\[
\xi^*(\bar{c}) = \frac{a - \bar{c}}{2 - \theta},
\]
\[
\xi^*(c) = \frac{2(a - \bar{c}) + (2 - \theta)(\bar{c} - c)}{2(2 - \theta)}.
\]

Now, consider the problem of firm \(i\), with \(i \in \{1, 2\}\), if it produces at date 1. Given the assumed strategy, firm \(j\), with \(j \neq i \in \{1, 2\}\), will produce \(\lambda^*(c)\) at date 1, if \(c_j = c\); if \(c_j = \bar{c}\), it will produce \(\lambda^*(\bar{c})\) at date 1, with probability \(\bar{\alpha}\), and \(\phi^*(q_i, \bar{c})\) at date 2, with probability \(1 - \bar{\alpha}\), after firm \(i\) has produced \(q_i\) at date 1. Therefore, firm \(i\) solves the problem:

\[
\max_{q_i} \quad \gamma (a - q_i + \theta \lambda^*(c) - c_i) q_i + (1 - \gamma)\bar{\alpha} (a - q_i + \theta \lambda^*(\bar{c}) - c_i) q_i + (1 - \gamma)(1 - \bar{\alpha}) (a - q_i + \theta \phi^*(q_i, \bar{c}) - c_i) q_i.
\]

This leads to the first-order conditions:

\[
a - 2\lambda^*(c_i) - c_i + \theta(\gamma \lambda^*(c) + (1 - \gamma)\bar{\alpha} \lambda^*(\bar{c}) + (1 - \gamma)(1 - \bar{\alpha}) \phi^*(\lambda^*(c_i), \bar{c})) - \frac{\theta}{2}(1 - \gamma)(1 - \bar{\alpha}) \lambda^*(c_i) \equiv 0, i \neq j \in \{1, 2\} \text{ and } c_i \in \{c, \bar{c}\}.
\]

This system of linear identities in \(\lambda^*(c_i)\), with \(i \in \{1, 2\}\), has a unique solution given by:

\[
\lambda^*(c) = \frac{(2 + (1 - \gamma)(1 - \bar{\alpha}) \theta)(2 - (1 - \gamma)(1 - \bar{\alpha}) \theta^2)(a - \bar{c}) + 2(2 - (1 - \gamma) \theta^2 - (1 - \gamma)(1 - \bar{\alpha}) \theta)(a - c)}{2(2 - (1 - \gamma)(1 - \bar{\alpha}) \theta^2)(2 - \gamma \theta - (1 - \gamma) \theta^2 - (1 - \gamma)(1 - \bar{\alpha}) \theta)}
\]
\[
\lambda^*(\bar{c}) = \frac{(2 + (1 - \gamma)(1 - \bar{\alpha}) \theta)(2 - (1 - \gamma)(1 - \bar{\alpha}) \theta^2)(a - \bar{c}) + 2(2 - (1 - \gamma) \theta^2 - (1 - \gamma)(1 - \bar{\alpha}) \theta)(a - c)}{2(2 - (1 - \gamma)(1 - \bar{\alpha}) \theta^2)(2 - \gamma \theta - (1 - \gamma) \theta^2 - (1 - \gamma)(1 - \bar{\alpha}) \theta)}
\]
As a result, if firm $i$ has cost equal to $\bar{c}$ and it produces at date 1, its expected profit will be:

$$\Pi_i^1(\bar{c}, \sigma^*) = \frac{(2 + (1 - \gamma)(1 - \hat{\alpha})\theta)(2 - (1 - \gamma)(1 - \hat{\alpha})\theta^2)}{8(2 - (1 - \gamma)(1 - \hat{\alpha})\theta^2)}(a - \bar{c}) + 2\gamma\theta(\bar{c} - c))^2,$$

while, if it produces at date 2, its expected profit will be:

$$\Pi_i^2(\bar{c}, s^*) = \gamma \left( \frac{(2 - (1 - \gamma)(1 - \hat{\alpha})\theta^2)(a - \bar{c}) + 2\gamma\theta(\bar{c} - c))^2}{4(2 - (1 - \gamma)(1 - \hat{\alpha})\theta^2)} \right)^2 + (1 - \gamma)\hat{\alpha} \left( \frac{(2 - (1 - \gamma)(1 - \hat{\alpha})\theta^2)(a - \bar{c}) + 2\gamma\theta(\bar{c} - c))^2}{4(2 - (1 - \gamma)(1 - \hat{\alpha})\theta^2)} \right)^2 + (1 - \gamma)(1 - \hat{\alpha}) \left( \frac{a - \bar{c}}{2 - \theta} \right)^2.$$

The probability $\hat{\alpha}$ will be such that a firm with cost equal to $\bar{c}$ is indifferent between producing at date 1 or producing at date 2. It is not possible to obtain a closed form solution for $\hat{\alpha}$, but numerical methods allow the conclusion that there is a unique value of $\alpha$ in $[0, 1]$ equalizing the two profits.

To conclude the proof, it should be checked that, under these conditions, if firm $i$ has cost equal to $c$, it will prefer to produce at date 1. Calculations lead to:

$$(\Pi_i^1(c, s^*) - \Pi_i^2(c, s^*)) - (\Pi_i^1(\bar{c}, s^*) - \Pi_i^2(\bar{c}, s^*)) = \frac{(1 - \hat{\alpha})(1 - \gamma)(2 - (1 - \gamma)(1 - \hat{\alpha})\theta^2)(a - \bar{c}) + (2 + \gamma\theta - (1 - \gamma)(1 - \hat{\alpha})\theta^2)(\bar{c} - c)^2}{4(2 - (1 - \gamma)(1 - \hat{\alpha})\theta^2)} \geq 0.$$

But, as $\hat{\alpha}$ is such that $\Pi_i^1(\bar{c}, \sigma^*) - \Pi_i^2(\bar{c}, \sigma^*) = 0$, the condition implies that $\Pi_i^1(c, \sigma^*) - \Pi_i^2(c, \sigma^*) \geq 0$. Hence, if firm $i$ has cost equal to $c$, it will prefer to produce at date 1.

This concludes the proof.
Proof of the Proposition 9

Suppose that there exists an equilibrium in which firm $i$ follows a strategy $\sigma_i^* = (\alpha_i^*, \lambda_i^*, \phi_i^*, \xi_i^*)$, with $\alpha_i^*(c) = 1$ and $\alpha_i^*(\bar{c}) = 0$, for $i \in \{1, 2\}$.

First, consider the problem of firm $i$, with $i \in \{1, 2\}$, at date 2, if it did not produce at date 1. If firm $j$, with $j \neq i \in \{1, 2\}$, has produced the quantity $q_j$ at date 1, firm $i$ will solve the problem:

$$\max_{q_i} (a - q_i - q_j - c_i) q_i,$$

which leads to the production of:

$$\phi^*(q_j, c_i) \equiv \frac{a - c_i - q_j}{2}.$$  \hfill (A46)

Note that the expression of the optimal response is independent of the firm’s identity.

If firm $j$, with $j \neq i \in \{1, 2\}$, has not produced at date 1, firm $i$ will infer that firm $j$ has cost equal to $\bar{c}$, and so firm $i$ will solve the problem:

$$\max_{q_i} \left( a - q_i - \xi_j^*(\bar{c}) - c_i \right) q_i,$$

which leads to the production of:

$$\xi_i^*(c_i) \equiv \phi^*(\xi_j^*(\bar{c}), c_i).$$

Considering the two firms and the two possible costs, the condition above gives raise to a system of four linear identities. This system has a unique solution, which is symmetric between the firms, i.e., for any $c \in \{c, \bar{c}\}$, $\xi_i^*(c) = \xi_2^*(c)$. Computing the solution one obtains:

$$\xi^*(\bar{c}) \equiv \frac{a - \bar{c} - \xi^*(\bar{c})}{2} \Leftrightarrow \xi^*(\bar{c}) \equiv \frac{a - \bar{c}}{3},$$  \hfill (A47)

and

$$\xi^*(c) \equiv \frac{a - c - \xi^*(c)}{2} \Leftrightarrow \xi^*(c) \equiv \frac{2(a - \bar{c}) + 3(\bar{c} - c)}{6}.  \hfill (A48)$$

Now, consider the problem of firm $i$, with $i \in \{1, 2\}$, if it produces at date 1. Given the assumed strategies, firm $j$, with $j \neq i \in \{1, 2\}$, will produce $\lambda_j^*(c)$ at date 1, if $c_j = c$, and it will
produce \( \phi_j^* (q_i, \bar{c}) \) at date 2, when \( c_j = \bar{c} \) and firm \( i \) has produced \( q_i \) at date 1. Therefore, firm \( i \) solves the problem:

\[
\max_{q_i} \text{Prob}\{c_j = c_i \} \left( a - q_i - \lambda_j^* (c_i) - c_i \right) q_i + \text{Prob}\{c_j = \bar{c} | c_i \} \left( a - q_i - \phi_j^* (q_i, \bar{c}) - c_i \right) q_i.
\]

This leads to the first-order conditions:

\[
a - 2\lambda_i^* (c_i) - c_i - \left( \text{Prob}\{c_j = c_i \} \lambda_i^* (c) + \text{Prob}\{c_j = \bar{c} | c_i \} \phi_j^* (\lambda_i^* (c_i), \bar{c}) \right) + \frac{1}{2} \text{Prob}\{c_j = \bar{c} | c_i \} \lambda_i^* (c_i) \equiv 0, \quad i \in \{1, 2\} \text{ and } c_i \in \{c, \bar{c}\}.
\]

This system of linear identities in \( \lambda_1^* (\cdot) \) and \( \lambda_2^* (\cdot) \) has a unique solution, which is symmetric (i.e., both firm use the same function \( \lambda^* \)). Solving it for each possible value of \( c_i \):

\[
a - 2\lambda^* (c) - c - \left( \frac{1 + \rho}{2} \lambda^* (c) + \frac{1 - \rho}{2} \phi^* (\lambda^* (c), \bar{c}) \right) + \frac{11 - \rho}{2} \lambda^* (c) \equiv 0
\]

\[
\Leftrightarrow \lambda^* (c) \equiv \frac{(3 + \rho)(a - \bar{c}) + 4(\bar{c} - c)}{4(2 + \rho)}, \quad (A49)
\]

and:

\[
a - 2\lambda^* (\bar{c}) - \bar{c} - \left( \frac{1 - \rho}{2} \lambda^* (\bar{c}) + \frac{1 + \rho}{2} \phi^* (\lambda^* (\bar{c}), \bar{c}) \right) + \frac{11 + \rho}{2} \lambda^* (\bar{c}) \equiv 0
\]

\[
\Leftrightarrow \lambda^* (\bar{c}) \equiv \frac{(9 + 4\rho - \rho^2)(a - \bar{c}) - 4(1 - \rho)(\bar{c} - c)}{4(2 + \rho)(3 - \rho)}. \quad (A50)
\]

So, it is concluded that, if an equilibrium with the assumed conditions exist, the two firms will follow the same strategy, i.e., \( \sigma_1^* = \sigma_2^* = \sigma^* \), which is unique.

The optimal moment of production is determined by comparing the associated expected profits:

1. When firm \( i \), with \( i \in \{1, 2\} \), has cost equal to \( c \):

   (a) If firm \( i \) produces at date 1, its expected profit will be:

   \[
   \Pi_i^1 (c, \sigma^*) = \frac{(3 + \rho)(3 + \rho)(a - \bar{c}) + 4(\bar{c} - c))^2}{64(2 + \rho)^2}.
   \]
(b) If firm \( i \) produces at date 2, its expected profit will be:

\[
\Pi^2_i(c, \sigma^*) = \frac{1 + \rho}{2} \left( \frac{(5 + 3\rho)(a - \bar{c}) + 4(1 + \rho)(\bar{c} - \bar{c})}{8(2 + \rho)} \right)^2 + \frac{1 - \rho}{2} \left( \frac{2(a - \bar{c}) + 3(\bar{c} - \bar{c})}{6} \right)^2.
\]

Comparing the two possible payoffs of firm \( i \), one obtains:

\[
\frac{(3 + \rho)(3 + \rho)(a - \bar{c}) + 4(\bar{c} - \bar{c})^2}{64(2 + \rho)^2} - \frac{1 + \rho}{2} \left( \frac{(5 + 3\rho)(a - \bar{c}) + 4(1 + \rho)(\bar{c} - \bar{c})}{8(2 + \rho)} \right)^2 - \frac{1 - \rho}{2} \left( \frac{2(a - \bar{c}) + 3(\bar{c} - \bar{c})}{6} \right)^2 = \frac{(1 - \rho)(1 - \rho)(5 + \rho)(a - \bar{c})^2 + 24(7 + 4\rho + \rho^2)(a - \bar{c})(\bar{c} - \bar{c}) + 144(\bar{c} - \bar{c})^2}{1152(2 + \rho)^2}
\]

which, given the restrictions on the parameters, is positive. So, firm \( i \)’s expected profit from producing at date 1 is greater than that from producing at date 2. Hence, \( \tau^i(c) = 1 \) (or \( \alpha^*(c) = 1 \)).

2. When firm \( i \) has cost equal to \( \bar{c} \):

(a) If firm \( i \) produces at date 1, its expected profit will be:

\[
\Pi^1_i(\bar{c}, \sigma^*) = \frac{(9 + 4\rho - \rho^2)(a - \bar{c}) - 4(1 - \rho)(\bar{c} - \bar{c})^2}{64(2 + \rho)^2(3 - \rho)}.
\]

(b) If firm \( i \) produces at date 2, its expected profit will be:

\[
\Pi^2_i(\bar{c}, \sigma^*) = \frac{1 - \rho}{2} \left( \frac{(5 + 3\rho)(a - \bar{c}) - 4(1 - \rho)(\bar{c} - \bar{c})}{8(2 + \rho)} \right)^2 + \frac{1 + \rho}{2} \left( \frac{a - \bar{c}}{3} \right)^2.
\]

Comparing the two expected profits of firm \( i \), one concludes that firm \( i \) will prefer to produce at date 2 if:

\[
\frac{(1 + \rho)(-15 + 91\rho + 37\rho^2 + \rho^3)(a - \bar{c})^2 + 72(1 - \rho)(3 + \rho)(a - \bar{c})(\bar{c} - \bar{c}) + 144(1 - \rho)(\bar{c} - \bar{c})^2}{1152(2 + \rho)^2(3 - \rho)} \geq 0.
\]

This condition is equivalent to:

\[
\frac{\bar{c} - \bar{c}}{a - \bar{c}} \geq \frac{2(123 + 19\rho + \rho^2 + \rho^3) - 48(2 + \rho)\sqrt{2(1 - \rho)(3 - \rho)}}{2(87 + 91\rho - 35\rho^2 + \rho^3)}.
\]

(A51)
For values of $\rho$ lesser than a negative value, determined numerically to be around $-0.178$, the right hand side of this condition is negative, and so the condition is trivially satisfied; for greater values of $\rho$, the bound is in $[0, 1]$, so that the equilibrium exists if condition (A51) is satisfied.

This concludes the proof.
Proof of Proposition 10

Suppose that there exists an equilibrium in which firm $i$ follows the strategy $\sigma^*_i = (\alpha^*_i, \lambda^*_i, \phi^*_i, \xi^*_i)$, with $\alpha^*_i(c) = 1$ and $\alpha^*_i(\bar{c}) = \hat{\alpha}$, with $0 < \hat{\alpha} < 1$.5

The problem of firm $i$, with $i \in \{1, 2\}$, at date 2 if it did not produce at date 1 is exactly the same of Proposition 9. Hence:

$$\phi^*_i(q_j, c_i) = \phi(q_j, c_i) = a - c_i - q_j$$

(A52)

$$\xi^*_i(\bar{c}) = \xi^*(\bar{c}) = \frac{a - \bar{c}}{3}$$

(A53)

$$\xi^*_i(c) = \xi^*(c) = \frac{2(a - \bar{c}) + 3(\bar{c} - c)}{6}.$$  

(A54)

Now, consider the problem of firm $i$, with $i \in \{1, 2\}$, if it produces at date 1. Given the assumed strategy, firm $j$, with $j \neq i \in \{1, 2\}$, will produce $\lambda^*_i(c)$ at date 1, if $c_j = c$; if $c_j = \bar{c}$, it will produce $\lambda^*_i(\bar{c})$ at date 1, with probability $\hat{\alpha}$, and $\phi^*(q_i, \bar{c})$ at date 2, with probability $1 - \hat{\alpha}$, after firm $i$ has produced $q_i$ at date 1. Therefore, firm $i$ solves the problem:

$$\max_{q_i} \text{Prob}\{c_j = c_i\} \left( a - q_i - \lambda^*_i(c) - c_i \right) q_i$$

$$\text{Prob}\{c_j = \bar{c} \hat{\alpha}\left( a - q_i - \lambda^*_i(\bar{c}) - c_i \right) q_i$$

$$\text{Prob}\{c_j = \bar{c} (1 - \hat{\alpha}) (a - q_i - \phi^*(q_i, \bar{c}) - c_i) q_i.$$

This leads to the first-order conditions:

$$a - 2\lambda^*_i(c_i) - c_i$$

$$- \left( \text{Prob}\{c_j = c_i\} \lambda^*_i(c) + \text{Prob}\{c_j = \bar{c} \hat{\alpha}\lambda^*_i(\bar{c}) + \text{Prob}\{c_j = \bar{c} \hat{\alpha}(1 - \hat{\alpha}) \phi^*(\lambda^*_i(c_i), \bar{c}) \right)$$

$$+ \frac{1}{2} \text{Prob}\{c_j = \bar{c} \hat{\alpha}(1 - \hat{\alpha}) \lambda^*_i(c_i) \equiv 0, \quad i \neq j \in \{1, 2\} \text{ and } c_i \in \{c, \bar{c}\}.$$

This system of linear identities in $\lambda^*_i(c_i)$, with $i \in \{1, 2\}$, has a unique solution given by:

\[\text{It is assumed the same probability that each firm produces at date 1 when it has a high cost.}\]
\[
\lambda^*(\alpha) = \frac{(3+\rho)(3-\rho)+2(3+\rho)(1+\rho)\hat{\alpha}+(1+\rho)(1-\rho)\hat{\alpha}^2)(a-\hat{\alpha})+4(3-\rho+2(1+\rho)\hat{\alpha})(\hat{\alpha}-\bar{\alpha})}{4((2+\rho)(3-\rho)+(5+5\rho+2\rho^2)\alpha+(1+\rho)(1-\rho)\alpha^2)}.
\]
\[
\lambda^*(\bar{\alpha}) = \frac{(9+4\rho-\rho^2+2(3+2\rho+\rho^2)\hat{\alpha}+(1+\rho)(1-\rho)\hat{\alpha}^2)(a-\hat{\alpha})-4(1-\rho)(\hat{\alpha}-\bar{\alpha})}{4((2+\rho)(3-\rho)+(5+5\rho+2\rho^2)\alpha+(1+\rho)(1-\rho)\alpha^2)}.
\]

As a result, if firm \(i\) has cost equal to \(\bar{\alpha}\) and it produces at date 1, its expected profit will be:
\[
\Pi_i^1(\bar{\alpha}, c^*) = \frac{(3-\rho+(1+\rho)\hat{\alpha})\left((9+4\rho-\rho^2)+2(3+2\rho+\rho^2)\hat{\alpha}+(1+\rho)(1-\rho)\hat{\alpha}^2)(a-\hat{\alpha})-4(1-\rho)(\hat{\alpha}-\bar{\alpha})\right)^2}{64((2+\rho)(3-\rho)+(5+5\rho+2\rho^2)\alpha+(1+\rho)(1-\rho)\alpha^2)^2},
\]
while, if it produces at date 2, its expected profit will be:
\[
\Pi_i^2(\bar{\alpha}, c^*) = \frac{1-\rho}{2} \left((35+3\rho)+(2(7+6\rho+3\rho^2)\hat{\alpha}+(1+\rho)(1-\rho)\hat{\alpha}^2)(a-\hat{\alpha})-4(3-\rho+2(1+\rho)\hat{\alpha})(\hat{\alpha}-\bar{\alpha})\right)^2 + \frac{1+\rho}{2} \left((35-\rho)+(2(7+6\rho+3\rho^2)\hat{\alpha}+(1+\rho)(1-\rho)\hat{\alpha}^2)(a-\hat{\alpha})+4(1-\rho)(\hat{\alpha}-\bar{\alpha})\right)^2 + \frac{1-\rho}{2} (1 - \hat{\alpha}) \left(\frac{2-\alpha}{3}\right)^2.
\]

The probability \(\hat{\alpha}\) will be such that a firm with cost equal to \(\bar{\alpha}\) will be indifferent between producing at date 1 or producing at date 2. It is not possible to obtain a closed form solution for \(\hat{\alpha}\), but numerical methods allow the conclusion that there is a unique value of \(\hat{\alpha}\) in \([0,1]\) equalizing the two profits.

To conclude the proof, it should be checked that, under these conditions, if firm \(i\) has cost equal to \(c\), it will prefer to produce at date 1. Calculations lead to [TO BE COMPLETED]:
\[
(Pi_i^1(\bar{\alpha}, c^*) - Pi_i^2(\bar{\alpha}, c^*)) - (Pi_i(\bar{\alpha}, c^*) - Pi_i^2(\bar{\alpha}, c^*)) = 0.
\]

But, as \(\hat{\alpha}\) is such that \(Pi_i^1(\bar{\alpha}, c^*) - Pi_i^2(\bar{\alpha}, c^*) = 0\), the condition implies that \(Pi_i^1(\bar{\alpha}, c^*) - Pi_i^2(\bar{\alpha}, c^*) \geq 0\). Hence, if firm \(i\) has cost equal to \(c\), it will prefer to produce at date 1.

This concludes the proof.
Proof of Proposition 11

Suppose that there exists an equilibrium in which firm $i$ follows a strategy $\sigma^*_i = (\alpha^*_i, \lambda^*_i, \phi^*_i, \xi^*_i)$, with $\alpha^*_i(\bar{c}) = 0$ and $\alpha^*_i(\bar{c}) = 1$, for $i \in \{1, 2\}$.

First, consider the problem of firm $i$, with $i \in \{1, 2\}$, at date 2, if it did not produce at date 1.

If firm $j$, with $j \neq i \in \{1, 2\}$, has produced the quantity $q_j$ at date 1, firm $i$ will solve the problem:

$$\max_{q_i} \left( a - q_i - q_j - c_i \right) q_i,$$

which leads to the production of:

$$\phi^*(q_j, c_i) \equiv \frac{a - c_i - q_j}{2}. \quad (A59)$$

Note that the expression of the optimal response is independent of the firm’s identity.

If firm $j$, with $j \neq i \in \{1, 2\}$, has not produced at date 1, firm $i$ will infer that firm $j$ has cost equal to $c$, and so firm $i$ will solve the problem:

$$\max_{q_i} \left( a - q_i - \xi^*_j(c) - c_i \right) q_i,$$

which leads to the production of:

$$\xi^*_i(c_i) \equiv \phi^*(\xi^*_j(c), c_i).$$

Considering the two firms and the two possible costs, the condition above gives rise to a system of four linear identities. This system has a unique solution, which is symmetric between the firms, i.e., for any $c \in \{c, \bar{c}\}$, $\xi^*_1(c) = \xi^*_2(c)$. Computing the solution one obtains:

$$\xi^*(c) \equiv \frac{a - \bar{c} - \xi^*(\bar{c})}{2} \Leftrightarrow \xi^*(c) \equiv \frac{a - c}{3}, \quad (A60)$$

and

$$\xi^*(\bar{c}) \equiv \frac{a - \bar{c} - \xi^*(c)}{2} \Leftrightarrow \xi^*(\bar{c}) \equiv \frac{2(a - \bar{c}) - (\bar{c} - c)}{6}. \quad (A61)$$

Now, consider the problem of firm $i$, with $i \in \{1, 2\}$, if it produces at date 1. Given the assumed strategies, firm $j$, with $j \neq i \in \{1, 2\}$, will produce $\lambda^*_j(\bar{c})$ at date 1, if $c_j = \bar{c}$, and it will produce
\( \phi_j^*(q_i, c) \) at date 2, when \( c_j = c \) and firm \( i \) has produced \( q_i \) at date 1. Therefore, firm \( i \) solves the problem:

\[
\max_{q_i} \text{Prob}\{c_j = c|c_i\} \left( a - q_i - \phi_j^*(q_i, c) - c_i \right) q_i + \text{Prob}\{c_j = \bar{c}|c_i\} \left( a - q_i - \lambda_j^*(\bar{c}) - c_i \right) q_i.
\]

This leads to the first-order conditions:

\[
a - 2\lambda^*_i(c_i) - c_i - \left( \text{Prob}\{c_j = c|c_i\} \phi_j^*(\lambda^*_i(c_i), c) + \text{Prob}\{c_j = \bar{c}|c_i\} \lambda_j^*(\bar{c}) \right) + \frac{1}{2} \text{Prob}\{c_j = c|c_i\} \lambda^*_i(c_i) \equiv 0, \quad i \in \{1, 2\} \text{ and } c_i \in \{c, \bar{c}\}.
\]

This system of linear identities in \( \lambda^*_2(\cdot) \) and \( \lambda^*_2(\cdot) \) has a unique solution, which is symmetric (i.e., both firm use the same function \( \lambda^* \)). Solving it for each possible value of \( c_i \):

\[
a - 2\lambda^*(\bar{c}) - \bar{c} - \left( \frac{1}{2} - \rho \phi^*(\lambda^*(\bar{c}), c) + \frac{1}{2} \rho \lambda^*(\bar{c}) \right) + \frac{1}{2} - \rho \lambda^*(\bar{c}) \equiv 0
\]

\[
\Leftrightarrow \lambda^*(\bar{c}) \equiv \frac{(3 + \rho)(a - \bar{c}) - (1 - \rho)(\bar{c} - c)}{4(2 + \rho)}, \quad \text{(A62)}
\]

and:

\[
a - 2\lambda^*(c) - c - \left( \frac{1}{2} + \rho \phi^*(\lambda^*(c), c) + \frac{1}{2} \rho \lambda^*(\bar{c}) \right) + \frac{1}{2} + \rho \lambda^*(c) \equiv 0
\]

\[
\Leftrightarrow \lambda^*(c) \equiv \frac{(9 + 4\rho - \rho^2)(a - \bar{c}) + (13 - \rho^2)(\bar{c} - c)}{4(2 + \rho)(3 - \rho)}, \quad \text{(A63)}
\]

So, it is concluded that, if an equilibrium with the assumed conditions exist, the two firms will follow the same strategy, i.e., \( \sigma_1^* = \sigma_2^* = \sigma^* \), which is unique.

The optimal moment of production is determined by comparing the associated expected profits:

1. When firm \( i \) has cost equal to \( c \):

   (a) If firm \( i \) produces at date 1, its expected profit will be:

   \[
   \Pi_i^1(\bar{c}, \sigma^*) = \frac{((9 + 4\rho - \rho^2)(a - \bar{c}) + (13 - \rho^2)(\bar{c} - c))^2}{64(2 + \rho)^2(3 - \rho)}.
   \]
(b) If firm $i$ produces at date 2, its expected profit will be:

$$
\Pi_i^2(\bar{c}, \sigma^*) = \frac{1 + \rho}{2} \left( \frac{a - c}{3} \right)^2 + \frac{1 - \rho}{2} \left( \frac{(5 + 3\rho)(a - \bar{c}) + 3(3 + \rho)(\bar{c} - \bar{c})}{8(2 + \rho)} \right)^2.
$$

Comparing the two expected profits of firm $i$, one concludes that firm $i$ will prefer to produce at date 2 if:

$$
\frac{(1 + \rho)((15 + 91\rho + 37\rho^2 + \rho^3)(a - \bar{c})^2 + 2(123 + 19\rho + \rho^2 + \rho^3)(a - \bar{c})(\bar{c} - c) + (87 + 91\rho - 35\rho^2 + \rho^3)(\bar{c} - c)^2)}{1152(2 + \rho)^2(3 - \rho)} \geq 0. \quad (A64)
$$

2. When firm $i$, with $i \in \{1, 2\}$, has cost equal to $\bar{c}$:

(a) If firm $i$ produces at date 1, its expected profit will be:

$$
\Pi_i^1(\bar{c}, \sigma^*) = \frac{(3 + \rho)((3 + \rho)(a - \bar{c}) - (1 - \rho)(\bar{c} - c))^2}{64(2 + \rho)^2}.
$$

(b) If firm $i$ produces at date 2, its expected profit will be:

$$
\Pi_i^2(\bar{c}, \sigma^*) = \frac{1 - \rho}{2} \left( \frac{2(a - \bar{c}) - (\bar{c} - c)}{6} \right)^2 + \frac{1 + \rho}{2} \left( \frac{(5 + 3\rho)(a - \bar{c}) + (1 - \rho)(\bar{c} - c)}{8(2 + \rho)} \right)^2.
$$

Comparing the two possible payoffs of firm $i$, one obtains:

$$
\frac{(3 + \rho)((3 + \rho)(a - \bar{c}) - (\bar{c} - c))^2}{64(2 + \rho)^2} - \frac{1 + \rho}{2} \left( \frac{(5 + 3\rho)(a - \bar{c}) + (1 - \rho)(\bar{c} - c)}{8(2 + \rho)} \right)^2 - \frac{1 - \rho}{2} \left( \frac{2(a - \bar{c}) - (\bar{c} - c)}{6} \right)^2 = \frac{(1 - \rho)((5 - 4\rho + \rho^2)(a - \bar{c})^2 - 2(79 + 52\rho + 13\rho^2)(a - \bar{c})(\bar{c} - c) - (19 + 100\rho + 25\rho^2)(\bar{c} - c)^2)}{1152(2 + \rho)^2} \quad (A65)
$$

Numerical simulations of expressions (A64) and (A65) for a grid of parameter values shows that both expressions are positive only for a limited set of parameter values with sufficiently low difference in the possible costs and quite negative correlated costs. For example, if $(\bar{c} - c)/(a - c) = 0.1$, $\rho$ can be no more than -0.6; but, if $(\bar{c} - c)/(a - c) = 0.1$, $\rho$ needs to be lesser than -0.9.

This concludes the proof.
**Equilibrium in the Example in Subsection 4.4**

We should check that the strategies and beliefs described in the result are an equilibrium of the game.

Suppose a firm with cost equal to 0 that has produced $\frac{10}{3}$ at date 1. Let us identify its optimal production at date 2.

If the other firm has produced the quantity $q \leq 2$ at date 1, it is believed to have cost equal to 2, and so both firms will believe that they have different types. Hence, at date 2 both firm will produce the quantities required to reach the productions associated with the Cournot equilibrium with full information, i.e., the firm with the low cost will produce $\frac{2}{3}$ to reach the total production of 4, and the firm with the high cost will produce $2 - q$ to reach the total quantity of 2.

If the other firm has produced the quantity $2 < q < \frac{10}{3}$ at date 1, it is believed to have cost equal to 2, and so both firms will believe that they have different types. Hence, the firm with the high cost will not produce at date 2, because its date 1 production already exceeds the production associated with the Cournot equilibrium with full information, and the firm with the low cost will produce according to its best reply to reach the total production of $5 - \frac{q}{2}$.

If the other firm has produced a quantity $\frac{10}{3} \leq q$ at date 1, it is believed to have cost equal to 0, and so both firm will know they have low costs. As the production associated with the Cournot equilibrium with two low cost firms is $\frac{10}{3}$, neither firm will produce at date 2.

Let us now move to date 1. If the other firm follows the strategy described, this firm does not gain anything from producing more than $\frac{10}{3}$ at date 1. It cannot either gain form producing less and being mistakenly believed to have a high cost, which can only increase the quantity produced by the other firm at date 2.

Therefore, a firm with cost equal to 0 can not gain by deviating form the strategy described in the result.

Suppose now a firm with cost equal to 2 that has produced $\frac{11}{4}$ at date 1. Let us identify its optimal production at date 2.

If the other firm has produced the quantity $q < \frac{10}{3}$ at date 1, it is believed to have cost equal to 2, and so both firms will believe that they have high costs. The production in the Cournot equilibrium with full information is then equal to $\frac{8}{3}$ and therefore this firm will produce nothing in date 2 (the other firm will have total production equal to $\max\left\{\frac{21}{8}, q\right\}$, where $\frac{21}{8}$ is its best reply.

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production to the production of $\frac{11}{4}$ by this other firm).

If the other firm has produced the quantity $\frac{10}{3} \leq q$ at date 1, it is believed to have cost equal to 0, and so both firms will believe that they have different types. Hence, the firm with the high cost will not produce at date 2, because its date 1 production already exceeds the production associated with the Cournot equilibrium with full information.

Let us now move to date 1. If the other firm follows the strategy described, the production of $\frac{11}{4}$ maximizes the expected profit of this firm:

$$0.5 \left( 10 - q - \left(5 - \frac{q}{2}\right) - 2\right) q + 0.5 \left( 10 - q - \frac{11}{4} - 2\right) q.$$  

The associated profit is equal to $\frac{363}{64}$ which is greater than the profit it would expect if producing $\frac{10}{3}$ in date 1 and being mistaken by a firm with a low cost, which is $\frac{65}{17}$.

Therefore, a firm with cost equal to 2 can not gain by deviating from the strategy described in the result.

Hence, the result follows.
References


