Optimal dynamic pricing for morning commute parking with occupancy information

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Abstract

Downtown parking is a challenging issue for both travelers and transportation planners. The objective of this paper is to maximize the benefits of parking management by optimal pricing and sensing. A generic parking model is presented for a set of sequential parking lots. Provided with the real-time parking information or with sufficient experience, travelers make parking location choices and/or departure time choices to minimize their generalized travel cost. We consider a general parking searching time function with respect to the occupancy. When the demand is inelastic in departure time, we show that the system optimum (SO) pricing solution is not unique. The non-uniqueness indeed offers much flexibility to set different dynamic pricing schemes for a variety of parking management goals. For the case where travelers have the flexibility to change their departure times, the SO is not feasible. We propose the minimum congestion (MC) pricing where the arrival rates to each lot exactly match the flow capacity. The model can be implemented in practice by utilizing parking sensors to set optimal occupancy-driven parking prices. It is found that the parking pricing and the provision of occupancy information jointly manage the traffic in an efficient way.

Keywords: Parking, Parking occupancy, dynamic pricing, System optimum

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1. Introduction

Downtown parking is a challenging issue for travelers, and it is also among the most common problems for transportation system planners, operators and regulators. The objective of this paper is to maximize the benefits of parking management by implementing pricing and sensing interactively. A generic parking model is presented and serves as the theoretical foundation for parking management systems to develop optimal real-time pricing and the best information provision strategy.

We have seen parking as an essential component of transportation system and infrastructure. Statistics show that a typical vehicle is parked 23 hours each day, and uses several parking spaces each week (Litman, 2011). What is even more surprising is that parking searching time represents up to 40% of the total travel time (Axhausen et al., 1994), mostly caused by the shortage of supplying most preferred parking spaces near the destination. Nevertheless, parking has gained less attention than the traditional focus on vehicular or roadway systems in managing traffic. The price, availability and accessibility, the three major components of parking facilities, could considerably influence travelers’ travel behavior, which mode to take, where to park and when to depart home. Therefore, parking can be used to efficiently manage the traffic. Additionally, parking pricing may be less controversial than roadway pricing, since most of the travelers have been used to pay a parking fee for their trips.

Pricing and sensing interactively balance the supply and demand of parking infrastructure. Sensing monitors real-time usage of parking facilities, and it is then translated into information provision and real-time pricing, both of which are adjusted in real-time to divert dynamic traffic. This paper investigates the interaction between real-time pricing and real-time sensing (particularly parking occupancy) in a dynamic system and proposes its optimal pricing schemes.

Both pricing and sensing have been widely applied in transportation management (e.g., Kwong et al., 2009; Calthrop et al., 2000), and either pricing or sensing in parking is not a novel idea in the literature. However, their relationships are not fully understood yet and they were usually investigated in the steady-state context. In fact, there are quite a few descriptive and empirical studies on parking policies and integrated parking management (e.g., Axhausen et al., 1994; Thompson et al., 1998; Vianna et al., 2004). Shoup (2005) particularly investigated policies on parking pricing. He suggested that, without pricing availability information, the most convenient parking
should be priced such that the occupancy averages 85-90%. Variable fees, namely higher rates during peak periods, is recommended.

Although qualitative guideline for parking management or pricing is available, theoretical studies on parking modeling are few in the literature. Several scholars analyzed the parking fee in the steady-state context. Bifulco (1993) introduced several parking types, fees and average walking times to the steady-state traffic assignment model so as to evaluate the efficacy of several regulatory parking policies in a network. Several others focused on the influences of a predetermined parking fee to simplified networks (e.g., Glazer, 1992; Verhoef et al., 1995). Arnott and Rowse (1999) developed a structural model of parking for a ring-road network. They assumed the parking lots are uniformly distributed on the ring-road, and thereafter derived the pricing schemes to achieve the system optimum. The optimal parking pricing for a set of fare zones was discussed by D’Acierno et al. (2006) in the sense of balancing the modal split between private cars and transit systems. Though those static models provide a basic idea of traffic congestion and network performance, they do overlook dynamic queuing of traffic flow and time-varying traffic patterns. Also, the availability and accessibility of parking spaces are not explicitly discussed in those models.

In addition, few theoretical parking models consider parking cruising time. Arnott and Rowse (1999) derived the expected parking time, driving time and cruising distance for searching available parking spaces in an ideal ring road network for the daily parking. Anderson and de Palma (2004) assumed the cruising time to find a parking space is the reciprocal of expected percentage of available spaces, and their results are intriguing: “cruising for parking congests both parkers and through traffic, the benefits from parking pricing are substantially reduced”. This results may lie on the steady-state assumption.

As for parking modeling in dynamic context, Arnott et al. (1991) are among the first to study the parking problem by dynamic user equilibrium. They assumed that the parking spaces are continuously distributed along the freeway near the CBD and the number of parking spaces per unit distance from the CBD is constant. Then they embedded the parking problem in the well-known morning commute model (Vickrey, 1969) to show that parking fee itself can be efficient in increasing social welfare, and a combination of dynamic road toll and dynamic parking fee can yield the system optimum that maximized social welfare. Such a setup for parking was further extended by Zhang et al. (2008) to derive the daily commuting pattern that combines
both the morning and evening commute. More recently, the parking capacity allocation, parking accessibility and parking fees are examined for a dynamic network with one roadway bottleneck and two parking areas (Qian et al., 2011, 2012). Optimal (coarse) parking fee, capacity allocation and parking access times are derived which altogether yield the minimum system cost and minimum traffic delay. However, the parking information is not used in the system optimum setting and cruising time is also not modeled.

Equipped with parking sensors, the technology of collecting parking occupancy information has been available for dynamic parking management systems in the last two decades. In some cities, travelers are provided with parking information in the form of variable message signs (VMS) or traffic radio. The information is usually regarding the number of available spaces in certain parking lots. However, in most cases, parking prices are unchanged from day to day. Although occupancy information in VMS could be used to optimize the parking reliability (Mei et al., 2012), several empirical studies and simulation show that the benefits of providing only parking availability information may be marginal in congested network conditions (e.g., Asakura and Kashiwadani, 1994).

In the literature, sensing and its contribution to traffic management is not fully valued yet. The real-time occupancy data could be analyzed to set up the real-time parking pricing, which is adapted to real-time demand and operates the system in the optimal way. More recently, new sensing technology prompts more refined parking information. In the SFPark project, sensors were installed in each of the 5,000 parking spots to track when and where exactly parking is available (SFMTA, 2011). Unfortunately, there is a lack of underlying theoretical parking model to provide real-time optimal pricing that fully utilizes the valuable parking data.

This paper investigates dynamic parking pricing scheme for a set of sequential parking lots. The optimal pricing scheme relies on a model of the relationship among parking occupancy, parking fee and travelers’ generalized travel cost in the dynamic context. Parking cruising time is an important part of the travel cost. The ultimate objective is to use pricing and sensing interactively to achieve a minimum system cost for travelers in the morning commute. The procedure can be implemented in practice by utilizing parking sensors to set optimal on-line parking pricing. Such a parking management system is expected to improve system efficiency, reduce cruising time, as well as to mitigate traffic congestion.

The paper is organized as follows. We first propose the basic model,
notations and set-ups in Section 2. Next, we solve the User Equilibrium (UE) and System Optimum (SO) flow patterns for case where all traffic demand have choices of parking locations, but with fixed departing schedules. Section 4 discusses the case with departure time choices. Both cases are summarized in Section 5. Numerical examples are provided in Section 6 using traffic data collected from the campus of Stanford University to illustrate the relationship between the optimal dynamic parking pricing and the travel profile. Finally, we conclude the paper in Section 7.

2. The model

In this section, we first state in details how pricing and sensing should operate interactively and efficiently to build a parking management system. Then, the parking network and model adopted in this paper are proposed.

2.1. Pricing and sensing

Pricing has been an efficient instrument to adjust travel demand. However, most of the current parking prices are not time-varying, and may not be adaptive to real-time traffic demand. On the other hand, the cutting-edge sensing technology makes it possible to obtain unprecedented parking information, including real-time occupancy of all lots, even with precise location of available spots. By providing this information to the travelers, unnecessary parking cruising for fully occupied lots can be reduced. However, this improvement could be marginal to the transportation system, if travelers are only aware of the parking usage information. This is because such information cannot predict the future usage precisely nor take actions on managing the traffic, and thus information provision alone is not efficient to reduce parking congestion, especially for those most congested lots. Pricing can be used to control the traffic efficiently, along with the information provision.

Here we propose a pricing-sensing loop control system for a set of parking lots in the transportation network. The idea is shown in Figure 1. At any time stage, the parking system takes measurement of the real-time parking usage. The parking data are then analyzed to forecast the future usage and demand, and determine the optimal real-time parking prices for each lot. Furthermore, the parking usage (existing and predictive) information and parking prices are sent to travelers through smart phones, websites, variable message signs, radios and so on. With real-time information, travelers make their decisions on which lot to choose and when to leave. In the next time
stage, the up-to-date parking usage is detected and reported to the parking system to update how users respond to the prices and information, which is considered to set the new optimal prices. Overall, parking pricing and sensing jointly serve as a dynamic stabilized controller of the traffic.

This paper assumes the travel behavior that every traveler minimizes their travel cost by fully accessible information or sufficient traveling experience. It is used to forecast the future usage, and set the real-time optimal parking prices.

2.2. Parking network and generalized travel cost

Suppose there are \( n \) parking lots for commuters to choose from. They are located on their way to the office sequentially. Each lot represents a parking structure (namely parking garage) or a block of on-street parking spots. Such a sequence of parking lots are defined as \( \{n, n-1, \ldots, 2, 1\} \) and illustrated in Figure 2.

![Figure 2: A sequence of parking lots](image)

There are in all \( N \) travelers who leave home for the office. Each traveler that arrives in the system chooses a parking lot \( i \) and then walks to the office. We use \( \lambda(t) \) to denote the traveler departure rate at time \( t \). Therefore, \( \int_0^T \lambda(t) dt = N \) where \( T \) is the commuting ending time. The composite travel time \( \tau_i = \tau_i^d + \tau_i^w \) of a traveler choosing lot \( i \) is defined as the sum of the driving time from his home to lot \( i \), \( \tau_i^d \), and his walking time to the office \( \tau_i^w \).
Due to the parking structure, \( \tau_1 < \tau_2 < \cdots < \tau_{n-1} < \tau_n \) and \( \tau_1^d > \tau_2^d > \cdots > \tau_{n-1}^d > \tau_n^d \). If no fees are imposed in the parking lots and users are provided real-time information that every parking lot has vacant spaces, then clearly lot 1 is the most desirable and lot \( n \) is the least desirable.

Each parking lot \( i \) has a capacity of \( K_i \) spaces. \( k_i(t) \) represents the number of occupied spaces in lot \( i \) at time \( t \). \( k_i(t) \leq K_i, \forall t \). Parking fee is represented by \( p_i(t) \) at time \( t \). In this paper, we consider the parking of the morning commuters. Therefore, the parking fee \( p_i(t) \) is one-time charge upon the traveler's arrival, and is independent of parking durations.

The generalized travel cost of travelers choosing parking lot \( i \) and departing home at time \( t \), \( C_i(t) \) (as \( t + \tau_i^d \) is his arrival time to lot \( i \)), consists of the parking fee, composite travel time and parking searching time,

\[
C_i(t) = p_i(t + \tau_i^d) + \alpha(\tau_i + f_i(k_i(t + \tau_i^d))) \tag{1}
\]

where \( \alpha \) is the average value of time (VOT) for the traveler population. \( f_i(t) \) denotes the expected parking cruising time for a traveler arriving in parking lot \( i \) at time \( t \). This time is in fact closely related to the parking occupancy and the type of parking information provided to the users. We will discuss more about the expected cruising time in a moment (later we use the phase “cruising time” or “searching time” directly for simplicity purposes).

Equation 1 can be simplified by adopting the departure time from the origin as the unified time line. It then reads,

\[
C_i(t) = \tilde{p}_i(t) + \alpha(\tau_i + f_i(\tilde{k}_i(t))) \tag{2}
\]

where \( \tilde{p}_i(t) \) and \( \tilde{k}_i(t) \) represents the parking price and the number of occupied spaces of lot \( i \) for a traveler departing home at time \( t \) (since the driving time \( \tau_i^d \) is fixed), respectively. By definition, \( \tilde{p}_i(t) = p_i(t + \tau_i^d) \), \( \tilde{k}_i(t) = k_i(t + \tau_i^d) \). Clearly, for the same traveler, his departure time from the lot is \( t + \tau_i^d + f_i(\tilde{k}_i(t)) \); his arrival time to the office is \( t + \tau_i + f_i(\tilde{k}_i(t)) \).

When a traveler has not only the choice of parking locations, but also the choice of departure times, we further introduce the schedule delay cost into the generalized travel cost, which now reads,

\[
C_i(t) = \tilde{p}_i(t) + \alpha(\tau_i + f_i(\tilde{k}_i(t))) + \beta(t^* - t - f_i(\tilde{k}_i(t)) - \tau_i) \tag{3}
\]

where \( t^* \) is the desired arrival time to the office (work starting time). A uniform work starting time, \( t^* \), is assumed. Here \( t^* - t - f_i(\tilde{k}_i(t)) - \tau_i \) is
the time one arrives earlier than the desired arrival time, also known as the schedule delay. $\beta$ is the values of schedule delay (VSD) of early arrival, which essentially converts the schedule delay to monetary cost. We assume no lateness is allowed.

Although the composite travel time is assumed to be constant, we relate the roadway congestion to the arrival rates to parking lots. The parking network in Figure 2 depicts the parking choices on arterials. The roadway congestion generated upstream of the parking network effectively determines the time-varying demand, i.e. $\lambda(t)$, which approximately equals to the bottleneck capacity. We wish to impose optimal parking pricing, so as to produce the desired demand profile $\lambda(t)$ over time that fits the bottleneck capacity. This can reduce the queuing delay caused by the bottleneck. The arrival rates to each parking lot can affect the roadway congestion as well. The on-street parking search creates additional congestion for through traffic; even for garage parking, overwhelming arrival rates can produce a long queue on the street due to limited service rate at the parking entrance. Therefore, we also wish to use parking pricing and information provision to allocate the arrival rate to each parking lot that matches the entrance capacity so that this type of roadway congestion can be eliminated.

2.3. Parking searching time

We assume a general family of searching time functions, $f_i(\tilde{k}_i(t))$. The expected searching time is dependent on the dynamic parking occupancy and how parking information is provided to the travelers. $\tilde{k}_i(t)$ is the occupancy occupied at departure time $t$. We use two types of occupancy later on, effective occupancy and nominal occupancy. Nominal occupancy is the number of spaces occupied at time $t$. However, nominal occupancy does not necessarily reflect the actual availability of the lot. Because by the time a vehicle arrives at the entrance of the lot, some travelers arriving earlier have been cruising around to find their spots, but not parked yet. Following the FIFO rule, a vehicle arrives at time $t$ find its spot later than all the vehicles arriving before $t$. The actual and meaningful occupancy for that vehicle is the cumulative arrived vehicles to that lot by time $t$. It is more reasonable to use the concept of effective occupancy $\tilde{k}_i(t)$, defined by the number of occupied spaces plus the number of cruising vehicles in lot $i$ by time $t$. $\tilde{k}_i(t) = \int_0^t \lambda_i(m) dm$. We will use the effective occupancy in the searching time function.

Now consider the case where travelers are provided with effective occupancy in each parking lot with, however, no exact spot location information
(this is the most common case in the current parking management system). A typical searching time function for travelers provided with overall parking occupancy information would look like Figure 3 (also reported by Axhausen et al., 1994; Horni et al., 2012). The function is increasing slowly under low or medium occupancy where parking time does not change much, because this is when finding an empty spot is relatively easy. However, the searching time increases dramatically when the occupancy is high, especially for the very last few spots. The cruising time is infinite when the lot is fully occupied. Thus, if a traveler is aware of high occupancy (e.g. 99%) and has no designated spot for him, he is unlikely to choose that lot due to high expected cruising time.

Such expected searching time function may only exist when all travelers using this lot have full access to the availability information (by online information system or day-to-day experience). When such information is not available, the cruising time may be rather random. Travelers may randomly choose one lot, cruise around a bit before he finds no vacancy and proceed to another lot (although there are vacant spots in the first one and they do not find them!). In fact, the real-time availability information ensures a stabilized parking choice in real time.

$$f_i(\tilde{k}_i(t)) = \frac{\varepsilon_i\delta}{1 - \frac{\tilde{k}_i(t)}{K_i}}$$ (4)

We here show a few special forms of the cruising time function. Axhausen et al. (1994) proposed an average searching time function by fitting it to real parking data provided with and without occupancy information,
where $\varepsilon_i$ is the average time spent on searching one parking space in an empty lot $i$. $\delta < 1$ is an adjustment factor to account for how the occupancy information is adopted by travelers. The function form with $\delta = 1$ has been discussed in the literature (Anderson and de Palma, 2004, e.g.). Axhausen et al. (1994) fitted the cruising time data using this function, resulting a very reasonable coefficient of determination $R^2 = 0.91$. This type of cruising time function is intuitive. By assuming that each parking spot in lot $i$ is equally likely to be vacant, the probability of each spot being vacant is $1 - \tilde{k}_i(t)/K_i$. Therefore, the expected number of parking spaces searched before locating a vacant space is $1/(1 - \tilde{k}_i(t)/K_i)$. We refer this type of searching function as “uniform” function later on.

The searching time function like Figure 3 may also be written as a BPR-like function, namely,

$$f_i(k_i(t)) = \varepsilon_i + \varepsilon'_i \left( \frac{k_i(t)}{K_i} \right)^r$$

(5)

where $\varepsilon'_i$ is a parameter that gives finite searching time for the last spot, rather than the infinite searching time in Equation 4. Usually $r$ needs to be greater than 4 in order to make the function resemble the shape in Figure 3.

In the case of an ideal parking system that assigns a spot for each traveler, then,

$$f_i(\tilde{k}_i(t)) = \left\{ \begin{array}{ll}
\gamma_i(\tilde{k}_i(t)/K_i) & \text{if } \tilde{k}_i(t)/K_i \leq 1 - \epsilon \\
\gamma_i(1 - \epsilon) + (\frac{\tilde{k}_i(t)}{K_i} + \epsilon - 1) \Delta & \text{if } \frac{\tilde{k}_i(t)}{K_i} > 1 - \epsilon 
\end{array} \right.$$ 

where $\gamma_i(z)$ is the driving time from the parking entrance to the $zK_i$-th parking spot in lot $i$. $\epsilon$ is a very small number close to 0, and $\Delta$ is sufficiently large. The lower case is only used to ensure the continuity of the function and the searching time goes to a sufficiently large number when the occupancy is 1.

Thereafter we assume the parking searching time function is strictly monotone with respect to the occupancy. This property ensures the uniqueness of the parking flow pattern.

2.4. Definition of User Equilibrium (UE) and System Optimum (SO)

We assume a User Equilibrium traveler behavior, where each of the travelers minimizes his own generalized travel cost and the cost eventually equalizes among all the travelers. UE essentially determines travelers’ choices of departure time and/or parking locations in real-time.
Definition of User Equilibrium (UE). Via either online parking information system or sufficient (and stabilized) traveling experience, travelers are aware of three types of information at any time, the real-time occupancy of each lot, the expected cruising time, and the real-time parking prices. Given the location, capacity and time-varying prices of all parking lots, the eventual flow patterns are such that, 1) the generalized travel costs of all travelers are the same; 2) No traveler can unilaterally change his parking choice and/or departure time choices to reduce his generalized travel cost.

The UE defined here is likely to occur in two situations. One is that after day-to-day experience, travelers are familiar with the time-of-time occupancy and thus the expected cruising time of those parking lots. The UE is clearly a day-to-day equilibrium. The other situation is that, travelers are also provided with the occupancy and estimated cruising time for each lot. When their parking choices are fully rational, i.e. they always choose the lot with the least generalized travel cost, a UE may also achieve. Any change in the pricing or parking facilities may lead to a new UE, but it takes some days to get stabilized for the former case, while the UE seems instantaneously achievable for the latter case. We are more interested in the latter UE, as this serves the foundation of analyzing travelers’ response to the online parking information.

Under UE behavior assumption, different parking prices lead to different UE flow patterns, and thus different network performance, represented by the total travel cost (including the cruising time, driving time and traffic congestion). The parking fees collected are generally not considered in the total cost. We are most concerned about the System Optimal (SO) pricing schemes that can lead to the minimal total cost. The resultant UE flow pattern under the SO prices is referred to as the SO flow pattern.

3. Inelastic demand without departure time choices

We first consider demand with parking location choices only. The time-dependent travel demand, i.e. \( \lambda(t) \) over \( t \), is predetermined, and travelers do not make departure time choices. This describes a population of inelastic demands, e.g., morning commuters with fixed departure schedules. In addition, if the time-varying demand to the lots can be measured accurately (using count sensors), this may be a proper model to show the UE parking flow pattern.
Given any time-varying prices \( p_i(t) \), we can easily solve the UE flow pattern by equalizing the generalized travel cost among those lots being used, at any time \( t \). Suppose \( m \) most preferred lots are used at time \( t \). Combining those \( m - 1 \) equations and the flow conservation equation yields the cumulative occupancy at each lot by time \( t \). If the parking price of each lot is constant, then the most preferred parking lot (with minimum fixed parking costs \( q_i = p_i + \alpha \tau_i + \alpha f_i(0) \)) is first used by travelers. The second preferred lot is not used until the searching time in the most preferred lot increases up to a certain level. The equilibrated travel cost goes up as time goes on until travelers start to use the third preferred lot, the fourth lot, and so on. By differentiating those \( m \) equations used to solve for UE, it is easy to show that,

\[
\frac{\partial \tilde{k}_i(t)}{\partial t} = \frac{\lambda(t)C}{\partial f_i(\tilde{k}_i(t))} C = \left( \sum_{i=1}^{m-1} \frac{1}{\partial f_i(\tilde{k}_i(t))} \right)^{-1}
\]

Equation 6 indicates that the arrival rate to each lot at any departure time is inversely proportional to its marginal cruising time increase rate. A lot with flatter cruising time function clearly gets more market share than a lot with rapid cruising time increase. Therefore, compared to the case where only occupancy information is provided to travelers, provision of vacant space locations (or maybe approximate space information) so as to reduce the searching time would certainly attract more travelers, therefore reduce the total cost.

### 3.1. SO Formulation

It is well known that SO achieves when the marginal travel cost (MTC) is equalized for any traveler departing at time \( t \) and choosing lot \( i \). Any additional traveler using lot \( i \), regardless of his departure time, can increase the total generalized travel time by his composite travel time \( \tau_i \) and the marginal parking cruising time \( f_i(\tilde{k}_i(T)) \). His contribution to the total cruising time is only dependent on the terminal occupancy, and is independent of his departure time.

Thus, the optimality condition for SO is,

\[
\tau_i + f_i(\tilde{k}_i^*(T)) = \frac{MTC}{\alpha}, \forall i = \{1, \ldots, m\}
\]
\[ \sum_{i=1}^{m} k_i^*(T) = N \] (7b)

The optimal terminal occupancy \( \{k_1^*(T), k_2^*(T), \ldots, k_m^*(T)\} \) is unique if the cruising time functions \( f_i(k), \forall i \) are strictly monotone. The optimality condition 7 can be solved efficiently by applying the bi-section search method.

3.2. Optimum parking pricing schemes

Although the terminal occupancy is unique, the arrival rates to the parking lots, however, are not unique under SO. Define \( \lambda \) as the vector of parking flow pattern, \( \{\lambda_1(t), \lambda_2(t), \ldots, \lambda_n(t)\}, \forall t \). Any parking flow pattern in the set \( \Omega = \{\lambda \geq 0 | \int_0^T \lambda_i(t) = k_i^*(T), \forall i, \sum_{i=1}^{m} \lambda_i(t) = \lambda(t), \forall t\} \) is optimum in terms of total cost. The non-uniqueness of the SO parking flow pattern indeed offers much flexibility to set different dynamic pricing schemes for a variety of management goals, while those pricing schemes could be SO. In fact, we can obtain any desired arrival rates to parking lots by setting appropriate SO pricing schemes. To see this, suppose the desired flow pattern is \( \bar{\lambda} = \{\bar{\lambda}_i(t), \forall i, t\} \in \Omega \). Let \( \bar{k}_i(t) = \int_0^t \bar{\lambda}_i(t')dt', \forall i, t \), which are desired time-varying effective occupancies. We use \( p_i^*(t) \) to represent the SO pricing scheme that realizes the flow pattern \( \bar{\lambda} \). We have,

\[ p_i^*(t) + \alpha(\tau_i + f_i(\bar{k}_i(t))) = p_j^*(t) + \alpha(\tau_j + f_j(\bar{k}_j(t))), \forall i,j \in S(t), \forall t \] (8)

where \( S(t) \) is the set of parking lots that being used at time \( t \). Differentiate both sides with respect to \( t \) and apply the chain rule, we have,

\[ \frac{\partial p_i^*(t)}{\partial t} + \alpha \frac{\partial f_i(\bar{k}_i(t))}{\partial k_i} \bar{\lambda}_i(t) = \frac{\partial p_j^*(t)}{\partial t} + \alpha \frac{\partial f_j(\bar{k}_j(t))}{\partial k_j} \bar{\lambda}_j(t), \forall i,j \in S(t) \] (9)

Equation 9 shows that if we set a constant parking fee for lot \( i \) in a short time period, we see that the change in the optimal price in lot \( j \) equals the difference in product of targeted flow rate and marginal searching time increase. A flatter searching time function of lot \( j \) (or a steeper searching time function of lot \( i \)) leads to a more dramatic change in the parking price of lot \( j \) that we need to achieve the system optimum. This has a strong policy indication. The parking price and the provision of occupancy information jointly serve as effective ways of managing traffic. The parking price and the provision of occupancy information should be balanced out. Given a targeted
parking flow pattern, great searching time change (i.e. high occupancy) in a lot, if provided to travelers, is a strong signal of not being able to find a spot, and thus requires a mild change in optimal parking price for that lot. With information provision, parking price is not the only economic manner to manage parking choices. Occupancy information will do the job too.

Given any desired SO parking flow pattern $\bar{\lambda}$, we can propose a corresponding SO pricing scheme to achieve it. This is done by solving Equation 8 sequentially for any time $t$. Note that we need to select a base lot in the set $S(t)$ where its price is set constant over time, and the prices of all the other lots are relative to this lot. Whenever the base lot is not used by any travelers under the targeted flow pattern, a new base lot should be selected.

3.3. Examples of parking management goals

The optimal terminal occupancy only ensures the minimum total travel time/cost, the network regulator are also concerned about the other factors, such as possible roadway congestion caused by cruising, queue spillback at the parking entrance. Those factors are not explicitly considered in the SO formulation, but their burden to the network performance can be minimized by carefully choosing an appropriate parking flow pattern $\bar{\lambda} = \{\bar{\lambda}_i(t), \forall i, t\}$.

1. Flow proportionality. One strategy that the network regulator may be interested in is to set the arrival rate to each parking lot proportional to its optimal terminal occupancy. Mathematically, it means,

$$\bar{\lambda}_i(t) = k^*_i(T) \frac{\lambda(t)}{N}, \forall i$$

In this way, travel demand at any time is directed to each lot in the same proportion as its final share in the total demand. It ensures each lot is filled up in approximately the same pace.

2. Inward parking preference. Parking prices are set such that travelers first choose to park in the farther area, and as times get close to the work starting time, travelers gradually transition to prefer the closer parking area. Some parking studies suggest that inward parking pattern is more efficient than outward parking pattern (e.g., Arnott et al., 1991; Qian et al., 2012).

3. Minimum congestion (Least Square Method). Parking could lead to additional traffic delay for travelers on the arterials. The delay may be caused by limited parking entrance service rate or limited capacity of
surface streets connecting to the parking lots. Either way, we would like to use an SO parking pricing to adjust the arrival rates for each parking lot such that minimum possible congestion occurs in both the parking area and the roadway. For each parking lot \( i \), the optimal final occupancy is given by \( k^*_i(T) \). The best arrival rate to lot \( i \) would be \( \lambda_i(t) = \frac{k^*_i(T)}{T} \). In that case, the traffic flow fully uses the roadway capacity and/or parking entrance capacity during the commuting period, and meanwhile the demand intensity is flattened for each lot as possible as it can. One common way is the least square minimization,

\[
\min_{\lambda \in \Omega} \sum_{i=1}^{m} \int_0^T \left( \lambda_i(t) - \frac{k^*_i(T)}{T} \right)^2 dt
\]  

which can be further simplified,

\[
\min_{\lambda \in \Omega} \sum_{i=1}^{m} \int_0^T \lambda_i(t)^2 dt
\]

4. Elastic demand with departure time choices

Now we consider the parking choices with departure time choices. This is true in reality when travelers are willing to make trade off between parking searching time (or fees if the garage provides early bird discount, for example) and departure times. Some travelers choose to leave home early so that they can easily find a parking spot, but they are subject to high penalty in early arrival to their offices. Other travelers may leave home late and thus their arrival time to the office can be very close to the targeted time, but clearly they have to spend more time on finding a vacant spot and/or paying more for parking. Throughout this section, the time-varying travel demand is no longer predetermined, but the total demand \( N \) is fixed. The time-varying travel demand, i.e. travelers’ departure time choice, along with their parking choice, is determined by the User Equilibrium condition.

4.1. UE profile

Under UE, for each of the lot, its last traveler arrives his office punctually. This is because if such a traveler does not exist, then the last traveler of using that lot, who now arrives the office a bit earlier then \( t^* \), can actually depart home a bit late such that he arrives punctually and therefore reduce the
schedule delay cost without changing the cruising time (also not interrupting any other travelers), which violates the UE condition.

Now suppose that \( m \) lots are used during the morning commute. We let \( \kappa_i \) denote the terminal occupancy in lot \( i \). Again, we assume the parking prices are fixed in this subsection for simplicity. According to the UE condition, the travel cost of the last traveler of each lot is identical. Because those travelers are not subject to schedule delay cost, we have,

\[
\frac{q_i}{\alpha} - f_i(0) + f_i(\kappa_i) = \frac{q_j}{\alpha} - f_j(0) + f_j(\kappa_j), \forall 1 \leq i < j \leq m, i, j \in \mathbb{N} \tag{13a}
\]

\[
\sum_{i=1}^{m} \kappa_i = N \tag{13b}
\]

Equation 13 can be solved efficiently using bi-section search for \( \kappa_i \) and \( m \).

According to the condition of UE that no traveler can change his parking departure time to reduce his travel cost, we have \( \frac{\partial C_i(t)}{\partial t} = 0 \), \( \forall i \in S(t) \).

Differentiate Equation 3 we have,

\[
\frac{\partial f_i(\tilde{k}_i(t))}{\partial t} = \frac{\partial f_i(\tilde{k}_i(t))}{\partial \tilde{k}_i} \frac{\partial \tilde{k}_i(t)}{\partial t} \Rightarrow \frac{\beta}{\alpha - \beta} \tag{14}
\]

This implies that under UE, if the prices do not change over time, the real-time occupancy for a parking lot is such that its cruising time increases linearly over time, and this increase rate \( \frac{\beta}{\alpha - \beta} \) is determined by the attributes of the traveler population.

For each lot \( i \) being used, given the terminal occupancy and Equation 14, we derive analytically travelers’ departure profiles. Let \( t_i \) represent the time when the first traveler use lot \( i \). Because the generalized travel cost of the first traveler using lot \( i \) is equal to that of the last traveler, we have

\[
p_i + \alpha(\tau_i + f_i(0)) + \beta(t^* - t_i - \tau_i - f_i(0)) = p_i + \alpha(\tau_i + f_i(\kappa_i)) \tag{15}
\]

Thus, \( t_i \) is,

\[
t_i = t^* - \tau_i - f_i(0) - \frac{\alpha(f_i(\kappa_i) - f_i(0))}{\beta} \tag{16}
\]

Travelers’ departure profile can thus be computed by Equation 14 with the boundary condition 16,

\[
\tilde{k}_i(t) = f_i^{-1}\left(\frac{\beta}{\alpha - \beta}(t - t_i) + f_i(0)\right) \tag{17}
\]
The UE solution is illustrated in Figure 4. The less fixed parking cost a lot has, the earlier travelers start to use it. If the cruising time function with respect to the occupancy is convex, then the departure curves are concave, meaning that the arrival rate to the lot decreases rapidly as the time gets close to the commute ending time.

![Figure 4](image)

Figure 4: The departure/arrival curve for lot $i$ under UE with departure time choices. 1. Departure curve from origin; 2. Arrival curve the lot; 3. Departure curve from the lot; 4. Arrival curve to the office

4.2. System Optimum solution

We first seek the optimal parking flow pattern that minimizes the total cost. The optimization problem reads,

$$\min_{\lambda} \ TC = \alpha \sum_{i=1}^{m} \left( \int_{0}^{\kappa_i} f_i(x)dx + \kappa_i \tau_i \right) + \beta \sum_{i=1}^{m} \int_{t_{i-1}}^{t_i} \left( t^* - t_{i-1} - \tau_i - f_i(\kappa_i) \right) dt$$

$$s.t. \sum_{i=1}^{m} \int_{0}^{t_{i-1}} \lambda_i(t)dt = N$$

(18a)

(18b)
\[ \kappa_i = \int_{0}^{t^* - t_1} \lambda_i(t) dt, \forall 1 \leq i \leq m, i \in \mathbb{N} \]  

(18c)

\[ \lambda_i(t) \geq 0, \forall 1 \leq i \leq m, 0 \leq t \leq t^* - t_1 \]  

(18d)

where \( \lambda \) is the vector of parking flow pattern, \( \{\lambda_1(t), \lambda_2(t), \ldots, \lambda_m(t)\} \), \( 0 \leq t \leq t^* - t_1 \). Here we set \( t_1 \), i.e. the departure time when the first traveler departs home, as the time origin. Any \( \lambda_i(t) \) and \( \tilde{k}_i(t) \) are with respect to the new time origin. In the objective function Equation 18, the first term is the total cruising time and total composite travel time, while the second term is the total schedule delay cost. Equations (or inequalities) 18b-18d are the feasibility constraints where the arrival rates to each lot should be nonnegative and the total demand equals to \( N \).

For any lot \( i \), given the number of spaces final occupied \( \kappa_i \), we should always shorten the arrival time window. The ideal case would be such that the schedule delay for any traveler is zero, namely everyone departs from the lot exactly at time \( t^* - \tau_i + \tau_i^d \). However this is not possible, since for any traveler departing at time \( t \) and \( t' \) and \( t < t' \), their arrival time to the lot becomes \( t^* - \tau_i + \tau_i^d - f_i(\tilde{k}_i(t)) > t^* - \tau_i + \tau_i^d - f_i(\tilde{k}_i(t')) \), which conflicts with \( t < t' \). Therefore, the minimum schedule delay cost is such that the arrival curve to the lot is vertical as depicted by the thick line (marked with SO) in Figure 5. In this way, travelers using the same parking lot will depart the origin at the same time, namely \( t^* - \tau_i - f_i(\kappa_i) \).

4.3. Minimum Congestion solution

Although a vertical arrival curve is truly the SO solution of optimization problem 18, it is far from being realistic. In fact, such an infinite arrival rate can lead to severe roadway congestion. A possibly best strategy of managing the flow pattern would be setting the arrival rate equal to the bottleneck capacity, denoted by \( \mu_i \). The capacity is either restricted by the parking entrance service rate or the capacity of the roadway near the parking lot \( i \).

It is then easy to see that the minimum SD achieves if the arrival curve to the lot has the rate \( \mu_i \) all the time. We call such a flow pattern the minimum congestion (MC) solution, as seen in Figure 5. Let \( t_i^* \) denote the departure time of the first traveler using lot \( i \) for the MC solution. Given \( \kappa_i \) for lot \( i \),

\[ t_i^* = t^* - \tau_i - f_i(\kappa_i) - \kappa_i \mu_i \]  

(19)
The last traveler arrives at the same time for both MC and SO

Figure 5: The SO and Minimum Congestion departure/arrival curves for lot $i$. 1. Arrival curve to the lot (MC); 2. Departure curve from the lot (MC); 3. Arrival curve to the lot (SO); 4. Departure curve from the lot (SO)

The minimum SD cost for lot $i$ with respect to $\kappa_i$, $SD^*_i(\kappa_i)$, becomes exactly the area below the departure curve from the lot (MC) in Figure 5, multiplied by the value of schedule delay $\beta$,

$$SD^*_i(\kappa_i) = \beta \left( \frac{\kappa_i^2}{2\mu_i} + \kappa_i f_i(\kappa_i) - \int_0^{\kappa_i} f_i(x)dx \right)$$  

(20)

Therefore, the MC solution can be formulated as,

$$\min TC = \alpha \sum_{i=1}^{m} \left( \int_0^{\kappa_i} f_i(x)dx + \kappa_i \tau_i \right) + \sum_{i=1}^{m} SD^*_i(\kappa_i)$$  

(21a)

s.t.  

$$\sum_{i=1}^{m} \kappa_i = N$$  

(21b)

$$\kappa_i \geq 0, \forall i \in \{1, \ldots, m\}$$  

(21c)

We obtain the optimality condition,

$$MTC = \alpha \tau_i + \alpha f_i(\kappa_i) + \beta \frac{\kappa_i}{\mu_i} + \beta \kappa_i f'_i(\kappa_i), \forall i \in \{1, \ldots, m\}$$  

(22a)
\[ s.t. \sum_{i=1}^{m} \kappa_i = N \quad (22b) \]

Given the properties of searching time function discussed in Section 2.3, the right hand side of Equation 22a is strictly monotone with respect to \( \kappa_i \). Therefore, the MC solution is unique and a bi-section search method can apply to solve the optimal \( \{\kappa_i\} \). Compared to the optimality condition 7 for the case where travelers’s departure times are fixed, we see that the lots being used under MC do not change with, however, different optimal terminal occupancies (as MTC has two extra terms here, and the both terms are zero when \( \kappa_i \) is zero). The reason is that if we assume travelers’ departure time changes in response to the price and parking information, then the MTC caused by any additional traveler is higher compared to the case with fixed departure times. The additional MTC is associated with the schedule delay cost, since any marginal traveler can increase the schedule delay of other travelers. For any lot \( i \), enlarging service rate \( \mu_i \) or reducing cruising time \( f'_i(\cdot) \) can attract more travelers under MC.

### 4.4. Optimal MC pricing

Once we obtain \( \kappa_i \) and \( m \) by solving Equation 22, we can construct the MC flow pattern \( \{k^*_i(t)\} \) for each lot \( i \) as described in Figure 5.

\[ k^*_i(t) = \mu_i(t - t^*_i), \quad \forall i, t^*_i < t \leq t^*_i + \frac{\kappa_i}{\mu_i} \quad (23) \]

Let \( t^*_i \) be the time that travelers start to use lot \( i \) under MC. For any lot \( i \) being used at time \( t \), its MC prices over time are \( (t^*_i < t \leq t^*_i + \frac{\kappa_i}{\mu_i}) \),

\[ p_i(t) = p_i(t^*_i) - (\alpha - \beta) \left( f_i(k^*_i(t)) - f_i(0) \right) + \beta \left( t - t^*_i \right) \quad (24) \]

We choose \( I = argmin\{t^*_i\} \), i.e. the lot which is used by travelers in the first place, and set the price of lot \( I \) at time \( t^*_I \) to \( p_0 \) as the base price.

For any lot \( i \) being used and \( i \neq I \), its price at time \( t^*_i \) is,

\[ p_i(t^*_i) = p_0 + (\alpha - \beta) \left( f_i(k^*_i(t^*_i)) - f_i(0) \right), \quad \forall i \neq I \quad (25) \]

Unlike the case where the time-varying demand is predetermined, the MC solution for flow patterns and pricing is unique if the cruising function is strictly monotone.
5. Discussion

If the demand is approximately inelastic in schedules, then the SO solutions minimizing the total cost are not unique. The terminal occupancy under SO is usually close to that under UE. Although the total cost reduction is not significant, the non-uniqueness of the SO parking flow pattern indeed offers much flexibility to set different dynamic pricing schemes for a variety of management goals, while all those pricing schemes could be SO. SO Prices can effectively shape the flow pattern to each lot. When the demand is fully elastic in departure schedules, the solution minimizing the total cost requires all the travelers to arrive simultaneously, which is not feasible. The best possible strategy that compromises the reduction of the searching time, schedule delay and roadway congestion is the MC solution, where each parking lot is priced at the level that the arrival rate is exactly the capacity flow rate. This is somewhat expected, and is analogous to the SO roadway roll in the classic morning commute model (Vickrey, 1969).

Occupancy-driven and time-varying parking pricing. We see that given the traveler population attributes, parking structure and their capacities, the SO or MC pricing scheme is solely dependent on the occupancies of all the lots over time. This has a strong policy indication for parking infrastructure management, that is, the time-varying parking fee for any lot \( i \) can be set in response to real-time occupancies, regardless of time dimension. If we have sensors deployed at each parking lot providing real-time occupancy data, then we are able to set the real-time pricing based solely on the occupancy of its own and/or adjacent lots. Time-varying parking prices can be used to achieve a variety of management objectives that lead the system to the optimum or close to the optimum.

A maximum occupancy cap should be imposed for each lot. For both inelastic and elastic demand, we conclude that there exists an optimal terminal parking occupancy for each of those lots. This essentially sets a targeted maximum parking occupancy by the end of the morning commute. If the central parking is underpriced (or even free), then the central parking lot is eventually used by more travelers than in the optimum case, and therefore subject to additional total system cost (especially additional cruising time). This is consistent with Shoup (2005)’s findings where he states the central parking should be priced to have certain percentage (e.g. 10%) of spaces available. Leaving some spots empty is beneficial for the system.

Save more convenient lots for peak traffic. For inelastic demand, compar-
ison between UE and SO (Equation 7) indicates that the most convenient lot, if priced to be used continuously throughout the entire commuting time, should always be under-priced at the end of the commuting time and over-priced in the beginning. Regardless of what SO flow patterns are targeted, eventually the farther lots are priced the same as the closer lots. Similarly, for elastic demand, it is even more beneficial to induce travelers to use the farther lots in the first place. This is because it can significantly shorten the schedule delay window and reduce the system cost. Either case, we need to identify the peak arrival time for parking demand, and price the most convenient lots high in off-peak to save them for peak traffic.

**Charge peak parking fee for elastic demand.** For elastic demand, the peak traffic is usually when most travelers depart at such time that the cruising time of preferred lots changes dramatically (refer to the medium-high occupancy in Figure 3). In order to balance the supply and demand for the peak time period, it is important to charge higher fee during that time. Outside of this peak period, lots can be under-priced to attract more travelers. *Early bird parking fee in less convenient lots is usually beneficial.*

**Avoid intensive arrival to one parking lot or area.** Under UE, it is usually the case that lots are filled sequentially one by one up to a high occupancy. We observe a long and small arrival tail at the end of the commuting time for the preferred lots, while in other time travelers arrive in a much higher rate than the flow capacity (roadway capacity or parking service capacity). This implies that the entering capacity for each lot or area is not fully utilized, not to mention possibly severe roadway congestion due to intensive arrival. Parking prices should be set to flatten the intensive arrival rates and all lots should be used in approximately the same pace over time.

**Choices of parking searching time function.** Various parameterized searching functions can result in different types of flow patterns. Possible choices of those functions are discussed in Section 2. If we have real-time occupancies for all lots, then this data would help us properly choose and calibrate the searching time functions. A general principle is that a slower occupancy increase in a lot usually indicates a faster searching time increase.

### 6. Numerical experiments

We are now interested in quantitatively identifying in the real cases the effectiveness of the SO pricing. The parking searching function will be calibrated in the real cases. Also, we compare the SO, MC and UE solutions in
terms of their resultant flow pattern, parking occupancies and the total cost.

We choose a real parking network on the campus of Stanford University. There are four major parking garages located on the (only) major arterial road from the western entrance of the campus. They are used by mostly university faculty, students and staff. There are a few spaces (less than 5%) deployed with meters that are specially used for visitors. Since the four parking lots are mostly for commuters, it is reasonable to assume all the travelers in this numerical experiment are aware of the approximate parking occupancy after their sufficient day-to-day experience. Therefore, UE flow pattern defined in Section 2 may exist. This will be further calibrated and validated using real traffic data.

Most of the travel demand head for the central part of the campus. This has been proven by recording license plate and identifying through traffic in several typical weekday mornings. The properties of those parking lots are listed in Table 1. The driving time and walking time is estimated approximately based on the distance measured in Google Earth. Currently, all lots are available for commuters with certain parking permits. The parking fees here are set to be consistent with average permit price on the daily basis. The VOT $\alpha$ is set to 35 US dollars per hour (approximately the average hourly pay rate on campus) and $\beta = \alpha/4$ by the literature (Arnott et al., 1990). We also collected 15-min traffic count data upstream of the inbound major arterial. The time horizon for this analysis is 7:00am-10:00am. The total travel demand is 1796 vehicles during the 3-hour peak period in a typical weekday.

<table>
<thead>
<tr>
<th>Table 1: The properties of the four parking lots</th>
</tr>
</thead>
<tbody>
<tr>
<td>Lot S-2</td>
</tr>
<tr>
<td>Driving time $\tau^d_i$ (Seconds)</td>
</tr>
<tr>
<td>Walking time (Seconds)</td>
</tr>
<tr>
<td>Parking capacity (spots)</td>
</tr>
<tr>
<td>Parking fee for UE case ($)</td>
</tr>
<tr>
<td>Calibrated $\varepsilon$ (Seconds)</td>
</tr>
<tr>
<td>$R^2$ in calibration</td>
</tr>
</tbody>
</table>

6.1. Calibration of the searching time functions

We first calibrate the searching time functions for each of the four parking lots using micro-simulation approaches. Here we adopt the form of uniform
cruising function as suggested by Axhausen et al. (1994) and Horni et al. (2012). For each lot, every spot is represented by its distance from the closest entrance. In the micro-simulation, vehicles are sent to the entrance sequentially. Each vehicle cruises for a spot with certain probabilities that are proportional to its distance. In addition, the cruising speed is assumed to inversely proportional to the visible occupancy. The cruising strategies follow Thompson and Richardson (1998) except no parking choices are involved in this model. We run the simulation for each lot for 50 days, and produce the cruising time of every single vehicle with respect to the effective occupancy upon its arrival. Those data points are then used to fit the uniform cruising time functions. \( R^2 \), Coefficient of determination, is used to determine the goodness of fit. The calibrated parameter \( \varepsilon \) and its corresponding (maximal) \( R^2 \) is also listed in Table 1. More than 80% of the variance in the simulated data can be explained by the calibrated cruising time functions.

The calibrated cruising time functions are then used to compute the UE and SO flow patterns. Since the 15-min traffic counts (i.e. traffic demand) are measured accurately, we use the model without departure time choices to validate the UE behavioral model. We plot the time-vary occupancy (both effective and nominal occupancy) against time in Figure 6.1. The most convenient lot, Lot S-2, is used in the first place. Lot 17 and 21 are not used until 8:00am when around 80% of the spaces in Lot S-2 has been occupied. Starting from 9:00am, travelers start to use the farthest lot. The terminal occupancy for the three closer lots is 92%, and the farthest lot is only used up to 77%. This is consistent with the actual traveler experience. The observed occupancy of one of the typical weekday at 7:30am, 8:30am and 9:30am for all four lots are marked in the figure. Overall, the resultant nominal occupancy matches the observation very well. The only exception is that our UE model over-estimates the occupancy by 25% for the Lot L-18/20 at 8:30am. This is possibly because we assume all the travelers go to the central campus, and this demand model does not account for the travelers heading for the offices near this farthest lot. In the future research, we shall model parking choices for multiple origins and multiple destinations to address this issue.

6.2. Value of sensing

Currently, no real-time information is provided to travelers on campus. The UE may exist because all travelers are familiar with the occupancy after sufficient commuting experience and day-to-day adjustment. We here
are interested in how information provision and commuting experience would change the system performance. Axhausen et al. (1994) showed that a smaller $\varepsilon$ implies less cruising time for an individual vehicle, provided with more parking occupancy information and more parking experience. We plot the changes in total cost and cruising time cost with respect to the parameter $\varepsilon$, for both cases with and without departure time choices (we assume the work starting time $t^*$ is 9:00am), as shown in Figure 7.

Compared to the baseline with the calibrated $\varepsilon$, if parking information provision can improve the efficiency of finding a spot by 50%, then the total system cost can be reduced by nearly 11% with inelastic demand and 21% with elastic demand. This contributes to a reduction of 48% and 35% in the total cruising time cost for both cases, respectively. Providing information, for example, the exact location of the vacant spots, can significantly benefit the system. The benefits are even more pronounced when the demand is flexible with departure time choices. Information provision also ensures a stabilized system.

6.3. **SO pricing without departure time choices**

Now we compare the UE solution and SO solution for the case of inelastic demand. Their terminal occupancies are shown in Table 2. In fact, the
difference in terminal occupancy is fairly small. The SO solution reduces the total cost by merely 0.37%.

The results are not surprising. The SO pricing directs several travelers to use the closer lots compared to UE solution, since those travelers can save a bit of searching time provided with occupancy information. When the travel demand is fixed over time (with little flexibility), it is certainly possible that the optimal pricing and information provision do not make significant difference in free-flow travel time and searching time. However, as we discussed before, SO pricing schemes for inelastic demand can effectively shape the flow pattern in any targeted rates, which is a desirable feature.

Table 2: The terminal occupancy under UE and SO with inelastic demand

<table>
<thead>
<tr>
<th></th>
<th>Lot S-2</th>
<th>Lot 21</th>
<th>Lot 17</th>
<th>Lot 18 &amp; 20</th>
<th>Total cost ($)</th>
<th>Reduction</th>
</tr>
</thead>
<tbody>
<tr>
<td>Capacity</td>
<td>736</td>
<td>188</td>
<td>505</td>
<td>608</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>UE</td>
<td>684</td>
<td>174</td>
<td>466</td>
<td>472</td>
<td>11,929</td>
<td>-</td>
</tr>
<tr>
<td>SO</td>
<td>698</td>
<td>177</td>
<td>469</td>
<td>450</td>
<td>11,885</td>
<td>0.37%</td>
</tr>
</tbody>
</table>

Now we investigate the SO parking flow pattern by implementing two different parking management strategies, inward parking and least-square minimum congestion. The time-varying SO parking fees and the resultant occupancy changes are shown in Figures 8.

For the inward parking strategy, we set the price such that the farther lot is preferred in the first place up to a light or medium occupancy level, and then its price gradually increases to favor closer lots. We set the time
Figure 8: The SO pricing and the resultant occupancy using inward parking (left: a, c, e) and minimum congestion strategy (right: b, d, f)
that travelers start to use the third, second and first closest lot to 8:17am, 8:32am and 9:06am. Then we see that the initial parking price for the four lots (from the closest to the farthest) is $8.5, $6, $4 and $0, considered as the early bird rates. Then as soon as a lot reaches its critical time, its price increases right away and some travelers start to use a closer lot. Finally they all increase to nearly $14 at 10:00am. The reason is because the inward parking inherently favors the outer lots in the first place, and the convenient lot is saved for peak traffic. Thus the parking prices for the closer lots should be set advantageous. The plot of parking price against the occupancy shows that the closest two lots should be regulated with almost fixed price, $13-$14 and $7, respectively. A high price of $14 should be set in the second closest lot, once it hits the occupancy of 0.95. The time-varying prices for the farthest two lots range $0-$12 and $4-$14, respectively, dependent on the real-time nominal occupancy.

The minimum congestion strategy produces the real-time occupancy that is almost proportional to the terminal occupancy at any time. When the occupancy is light to mild, the parking fee for the four lots are $8, $5.5, $3 and free from the closest to the farthest. Similar to the inward parking strategy, for each lot, the parking prices should be raised quickly once the occupancy is beyond a certain level.

6.4. MC pricing with departure time choices

Now we consider the case with elastic demand. The terminal occupancy under UE and MC solutions are shown in Table 3. For the MC solution, a capacity rate of one vehicle per second is adopted here. The time-varying occupancies and parking prices for both solutions are shown in Figure 9.

<table>
<thead>
<tr>
<th>Lot</th>
<th>Total cost ($)</th>
<th>Reduction</th>
</tr>
</thead>
<tbody>
<tr>
<td>S-2</td>
<td>680</td>
<td>-</td>
</tr>
<tr>
<td>Lot 21</td>
<td>172</td>
<td>-</td>
</tr>
<tr>
<td>Lot 17</td>
<td>472</td>
<td>-</td>
</tr>
<tr>
<td>Lot 18 &amp; 20</td>
<td>472</td>
<td>-</td>
</tr>
<tr>
<td>Total cost ($)</td>
<td>19,994</td>
<td></td>
</tr>
<tr>
<td>Reduction</td>
<td>-</td>
<td>28%</td>
</tr>
</tbody>
</table>

The MC solution tends to allocate more travelers to the farther lots, compared to UE. The order of the lots being used under MC is the the first, second, fourth and third farthest lots, while UE favors the first and third closest lots in the first place. The total cost is reduced under MC by $5,677,
a 28% reduction. The MC pricing and information provided is considered fairly effective in this case. By comparing the two flow patterns of the two solutions, we find that the UE produces intensive arrival rates to the lots (e.g., around 8:00am and 8:15am), and thus is generally not favored, while the MC solution directs travelers in a way that their arrival rates to each lot exactly match the capacity flow rate. More importantly, MC shortens the commuting time window from the starting time of 7:57am under UE to that of 8:27am. This can significantly reduce the schedule delay cost.

We find that the optimal terminal occupancy for the closest three lots is around 90.5%. This result is consistent with previous studies that suggests a 10% parking vacant spots in the congested area (Shoup, 2005). In other words, if the traffic demand is flexible in choose their departure times, a few vacant spots in the central parking area can benefits the system. Such an optimal terminal occupancy balances out the total searching time, free-flow travel time, schedule delay cost as well as roadway congestion, which is generally desirable in reality.

The time-varying pricing and occupancies for the MC solution are shown in Figure 10. Without loss of generality, the farthest lot is set to be free. The prices of the other three lots start with $9, $8, and $3.5, respectively. The corresponding MC prices increase to $12, $10, and $6 during 8:35am to 8:45am, when is exactly the time the two most preferred lots start to be used by travelers. The increase in price is due to flattening the intensive arrival curve of the peak traffic demand shown in the UE case, while the decrease
is to induce travelers to use lots when it is generally being under-used under UE. The peak period for the closest lot is the longest, from 8:32am to 8:48am.

![Parking price vs. Time](image1)

![Parking price vs. occupancy](image2)

Figure 10: The time-varying occupancy and parking prices under MC with schedule delay

7. Conclusions

This paper investigates dynamic parking pricing scheme and provision of parking information to travelers for a set of sequential parking lots. Travelers are either familiar with the parking facilities, or they can obtain the real-time parking occupancy information and pricing information for all the available parking lots. Then they make departure time choices and/or parking location choices to minimize their generalized travel cost. We consider a general parking searching time function as part of the travelers’ generalized travel cost. We first derive the parking flow pattern under the UE conditions for the cases with and without departure time choices, and then further solve the optimal parking prices and corresponding parking flow patterns. Policy indications for pricing and information provision are discussed.

Sensing in parking system provides real-time parking information to travelers, and thus travelers are less likely to randomly choose one lot or cruise around several parking lots. PARKING information indeed helps travelers estimate the searching time more precisely, effectively reduce the cruising time within each lot, and ensures a stabilized parking choice in real time. More importantly, it undoubtedly mitigates both parking congestion and roadway congestion so as to reduce the social costs and emissions.
Dynamic parking pricing is crucial in parking management. Our study show that it can not only effectively reduce the social costs, but also shape desired travel profiles. A more general setting of parking pricing includes both the occupancy-driven parking price and the provision of occupancy information, which jointly serve as effective ways of managing the traffic. The parking price and the provision of occupancy information should be balanced out. Given a targeted parking flow pattern, greater searching time change or greater occupancy change in a lot, if precisely estimated and provided to travelers, is also a strong signal of alerting travelers to avoid congestion. With information provision, parking price is not the only economic manner to manage parking choices. Parking information will do the job too, thanks to the sensing technology.

Although our model reveals the relationship of parking pricing, information provision and the parking flow pattern, we do make several simplifying assumptions in our models. First, parking choices and departure time choices are never deterministic, especially when real-time parking information is provided to travelers. Therefore, it would be of particular interest to model the stochasticity of the parking choices and solve for the system optimum in the stochastic cases; Second, we here assume homogeneous traveler population in the sense that their value of time, value of SD and their utility function (i.e. generalized travel cost function) are identical. This may be a strong assumption and could distort the fine results. We expect to extend our research in these two aspects.

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References


Optimal Parking Pricing in General Networks with Provision of Occupancy Information

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Abstract

This paper investigates the relations between dynamic parking prices and provision of parking information in a general parking network. Travelers are provided with the real-time occupancy and pricing information to make their parking choices. We first formulate the parking choices under the User Equilibrium (UE) conditions using the Variational Inequality (VI) approach. More importantly, the system optimal (SO) parking flow pattern and SO parking prices are also derived and solved efficiently using Linear Programming. Under SO, any two parking lots cannot be used at the same time by travelers between more than one O-D pairs. The SO parking flow pattern is not unique, which offers sufficient flexibility for operators to achieve different management objectives while keeping the flow pattern optimal. We show that any optimal flow pattern can be achieved by lot-based parking pricing schemes that only depend on the time or real-time occupancy. We finally solve both UE and SO in two numerical examples. The best system performance is usually achieved by the parking prices such that the more preferred (convenient) lot should be used fully up to a certain terminal occupancy of around 85%-95%. This essentially balances the parking congestion (namely cruising time) and the convenience of preferred lots. We also obtain the SO prices from the SO solution set, to produce constant arrival rates to each lot. This could mitigate the potential roadway congestion and queuing comparing to intensive arrival rates.

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Keywords: Parking management, optimal parking pricing, parking information provision, parking occupancy, parking cruising time

1. Introduction

The increasing demand of motorized vehicle and limited public space prompt severe parking issues in modern cities. Parking can considerably influence travel behavior, and thus is one of the urban problems that are top priorities for transportation planners. The objective of this paper is to improve parking management by imposing optimal parking prices and providing parking information.

Parking plays a pivotal role in transportation industry and life quality for travelers. Studies show that a typical car trip (if paying tolls or parking fees) pays parking fee for more than $5 on average, almost 70% of the direct travel cost (Vuchic, 1999). The parking prices, availability and accessibility, the three components

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of parking facilities, considerably influence travelers’ decisions, when to leave, which mode to choose and where to park. All those parking components should be optimized to efficiently manage the traffic so as to build a sustainable transportation system. Undoubtedly, given the limited land-use for parking spaces in most mega-cities, optimal parking pricing can be a flexible and desirable tool for most system planners and regulators.

Either parking pricing or information provision alone has been discussed intensively in the literature, but their relationships are not fully understood yet. Most of the parking studies are descriptive and provide qualitative guidelines for parking pricing in the steady-state context (e.g., Axhausen et al., 1994; Thompson and Richardson, 1998; Viana et al., 2004; Shoup, 2005; Litman, 2011). However, real-time dynamic pricing in parking, one of the efficient economic tools to balance real-time supply and demand, has gained less attention than traditional real-time roadway tolls. This is partially due to the difficulty of acquiring real-time supply data (namely parking information) and possible objection from travelers for not being able to obtain such real-time information. However, the cutting-edge sensing and information technology make it possible to obtain unprecedented information to overcome this difficulty. Equipped with parking sensors, modern parking lots or blocks can provide real-time usage of parking facilities. The parking information, prices and availability information for instance, can be sent through smartphones and GPS receivers to travelers in real time. We certainly expect sensing and information provision help build more efficient parking pricing policies, as well as to resolve travelers’ parking issues.

Parking information provision has been implemented in the last two decades, but it alone is not working effectively as expected in many cases. In some cities, travelers are provided with parking information in the form of variable message signs (VMS) or traffic radio broadcast. The real-time parking information is usually regarding the number of available spaces in certain parking lots or blocks. Although such general occupancy information in VMS may be used to optimize the parking reliability (Mei et al., 2012), both empirical studies and simulation show that the benefits of providing availability information alone may be marginal in congested network conditions (e.g., Asakura and Kashiwadani, 1994). This is essentially because when the parking spaces are scarce, the information alone does not deviate travelers from cruising for parking (i.e., parking congestion), nor can it considerably change travelers’ choices in departure time, routes or traffic modes. Clearly, this can certainly be improved by pricing. Therefore, it is essential to combine both dynamic pricing and sensing in parking management, and pricing and information provision jointly serve as the efficient economics leverage.

Theoretical parking modeling is rare in the literature, and most of those models are in the steady-state context or based on simplified (one-origin-one-destination or single roadway path) networks. Glazer (1992) investigated two different parking schemes, lump-sum parking fee and a fee per unit time, with their affect to the social welfare. Verhoef et al. (1995) conducted diagrammatic analysis on how parking affects the individual travel cost and modal split. Instead of assuming fixed parking demands and constant travel cost for all the travelers, Arnott and Rowse (1999) derived the optimum parking prices for a ring-road network (also parking facilities) by adopting uniformly distribution parking demand. The optimal parking pricing was further discussed by D’Acierno et al. (2006, 2011) between private cars and transit systems. Bifulco (1993) modeled parking pricing for general transportation networks. He incorporated several parking types and fees into the static traffic assignment for general networks. The efficacy of parking policies is evaluated for long-term planning purposes.

As for parking modeling in the dynamic context, Arnott et al. (1991) were among the first to study dynamic user equilibrium with respect to parking choices. A special network is assumed where parking spaces are continuously distributed along the (only) freeway connecting to the destination. The system dynamics were embedded in the bottleneck model (Vickrey, 1969), a single-origin-single-destination network, to show that parking fee alone can be efficient in increasing social welfare (but not to the extent of roadway tolls). This setup was further extended by Zhang et al. (2008) to include evening commute into consideration. More recently, all three parking components, availability, accessibility and prices are examined based on the bottleneck model where only two parking clusters are considered (Qian et al., 2011, 2012). However, the provision of real-time parking information is not part of the system optimum setting and parking congestion (parking searching time) is not modeled. In addition, parking cruising time has been calibrated using real data (Axhausen et al., 1994) and simulation studies (Gallo et al., 2011).
There is a great need to study the parking pricing and sensing for a general network, so that the optimal parking pricing strategy can be set in coordination with the information provision in the real world. In fact, a recent parking project in the City of San Francisco has started its exploration of optimal parking prices by analyzing parking occupancy (SFMTA, 2011). In the SFPark project, sensors were installed in each of the 5,000 parking spots to track when and where exactly those parking spaces are available. Wherever the monthly aggregated parking occupancy increases (or decreases), the parking rate will be adjusted to increase (or decrease) by 25 to 50 cents accordingly. Unfortunately, there is a lack of underlying theory for general networks to connect real-time parking prices with the occupancy.

This paper models travelers’ parking choices by assuming that travelers are provided with precise parking occupancy information and real-time parking prices. Qian and Rajagopal (2012) investigated the relations between parking pricing and real-time parking occupancy based on User Equilibrium. The optimal parking prices were further obtained in a close form for single-origin-single-destination traffic demand. This paper adopts a similar setting for modeling parking choices as in Qian and Rajagopal (2012), but considers general multi-origin-multi-destination traffic demands, and a more general pricing strategy. Each parking lot/block could be used by traffic demands associated with any origins and/or destinations. We explicitly model the dynamic parking prices and cruising time as part of the generalized travel cost. Both the information provision and parking prices jointly serve as the efficient way of managing the traffic. We obtain the optimal dynamic parking pricing to minimize the total system cost. As a result, the optimal parking prices with respect to the real-time occupancy can be obtained. This model can be implemented in practice for a parking management system where the real-time occupancy is acquired from sensors and the prices are adjusted in real-time. Such a system is expected to improve the efficiency of parking facilities, reduce parking congestion, as well as to mitigate traffic congestion.

The rest of the paper is organized as follows. We first present the basic model set-up and notations for the parking network in Section 2. We then formulate the parking choices using User Equilibrium in Section 3 and solve it through Variational Inequality approach. It is followed by solving the System Optimum (SO) flow patterns in Section 4. We derive the optimality condition for a general network show that the SO solution is not unique. A variety of parking control strategies can be embedded in the SO, and each is realized by a certain SO pricing scheme. In section 5, two numerical experiments are provided, a network with two origins and two destinations, and a real parking network of Stanford University campus. We conclude the paper in Section 6.

2. The model

In this section, we present the general network and travelers’ generalized travel cost. We also discuss the formula of parking searching time (also known as cruising time) and how it is embedded in our model.

2.1. The roadway network and model set-up

Suppose there are \( n \) parking lots/blocks in the network, numbered as \( \{1,2,\ldots,n-1,n\} \) and illustrated in Figure 1. Each of the parking lots/blocks represents a parking garage (structure) or a block of on-street parking spaces. Those parking lots/blocks are connected to the automobile network by single roadway links (the solid lines), or a series of roadway links and intersection nodes (the dash lines). Each parking lot \( i \) has a capacity of \( K_i \) spaces and its parking fee is \( p_i \). We discretize the time horizon into \( T \) time periods, \( \{1,2,\ldots,T\} \). Parking fee is dynamic and represented by \( p_i(t) \) at time \( t \). In addition, \( k_i(t) \) represents the number of effectively occupied spaces in lot/block \( i \) at time \( t \) (to be elaborated later). \( k_i(t) \leq K_i \), \( \forall i \in \{1,2,\ldots,n\}, \forall t \in \{1,2,\ldots,T\} \).

In transportation planning and network analysis, it is customary to use centroid nodes to represent traffic analysis zones (TAZ), from/to which trips are assumed to have originated (Origins) or destined (Destinations). In the network of Figure 1, suppose there are \( |\mathcal{R}| \) origin nodes and \( |\mathcal{S}| \) destination nodes where \( \mathcal{R} \) and \( \mathcal{S} \) are the set of origin nodes and destination nodes, respectively. Each traveler departs an origin, chooses a parking lot to park, and then walks to the destination. We use \( \lambda_{rs}(t) \) to denote the traveler departure rate (i.e. traffic demand) at a departure time \( t \) for an O-D pair \( r-s \), \( \forall r \in \mathcal{R}, \forall s \in \mathcal{S}, \forall t \in \{1,2,\ldots,T\} \). The traffic
demand is assumed to be pre-determined throughout this paper. \( \lambda_{rs}(t) \) denotes the traffic demand departing the origin \( r \) at time \( t \) heading for destination \( s \) and choosing the parking lot/block \( i \), \( \forall r \in R, \forall s \in S, \forall i \in \{1, 2, \ldots, n\} \). The vector of \( \lambda_{rs}(t) \) reads \( \Lambda = \{\lambda_{rs}(t)\}_{i,r,s} \), also known as the parking flow pattern.

2.2. The general parking network

We assume that travelers departing from the same origin (or arriving the same destination) and using the same parking lot choose the same roadway route (which is independent of the roadway congestion). Therefore, the roadway network of Figure 1 can be easily transformed into the graph shown in Figure 2 to represent the general parking network. The new graph consists of three disjoint node sets, \( R, S \) and \( \{1, 2, \ldots, n\} \), representing the set of origin nodes, destination nodes, and parking lots/blocks. Each of the origins is connected to all the lots with driving links. Similarly, each destination is connected from all the lots by walking links. The driving time from an origin node to any parking lot \( i \), \( \tau_{Ori} \) can be approximated by finding the minimum travel cost between node \( r \) and \( i \) in the roadway network. On the other hand, the walking time from a parking lot \( i \) to a destination node \( s \), denoted by \( \tau_{Dis} \), is approximately equal to the distance between the two nodes divided by the average walking speed.

We do not explicitly model roadway congestion in terms of travel time in this paper. The roadway travel time, i.e., driving time and walking time, is assumed to be relatively constant. Rather, we will be focusing on the minimization of the total cost consisting of parking cruising time and the constant travel time. This simplifies the parking network, and thus can yield the analytical solution for the system optimum for insights. The analytical solution allows us to explore the relations between parking pricing/information provision and travelers’ parking choices. The assumption is particularly reasonable when the general parking network in Figure 2 describes parking spots connected by local arterial roads. Here the parking lots are not far from travelers’ destination. In such a setting, travelers usually make parking choices when they are off the freeway. It is then reasonable to assume that travelers with the same origin are subject to the same traffic congestion on the freeway, and their driving time on the arterials to each parking lot/block is constant in a relative sense.

Although the roadway congestion is not modeled in driving time, we are concerned about the arrival rates to each lot which potentially affect the roadway congestion. The intensive parking demand searching for on-street parking creates additional congestion for through traffic; even for garage parking, overwhelming arrival rates can produce a long queue on the street due to limited service rate at the parking entrance. Therefore, we wish to use parking pricing and information provision to produce mild parking arrival rates, so that this type of roadway congestion can be eliminated. The arrival rates to each lot indeed can be controlled by the optimal parking pricing strategies.
2.3. Generalized travel cost

Now we define the generalized travel cost for travelers. They choose a parking lot by minimizing their generalized travel cost. The composite travel time $\tau_{rs}^i$ of a traveler is defined as the sum of the time from his origin $r$ to lot $i$ and his walking time to the destination node $s$. Thus,

$$\tau_{rs}^i = \tau_{ri}^O + \tau_{is}^D$$

If no fees are imposed in the parking lots and users have real time information about vacant spot locations, then clearly the lot closest to $s$ is the most desirable for any travelers heading for $s$.

The generalized travel cost of travelers between O-D pair $rs$ choosing parking lot $i$ and arriving at the lot at time $t$, $C_{rs}(t - \tau_{rs}^O)$ (as $t - \tau_{rs}^O$ is his departure time from $r$), consists of the parking fee, composite travel time and parking searching time,

$$C_{rs}(t - \tau_{rs}^O) = p_i(t) + \alpha(\tau_{rs}^i + f_i(k_i(t)))$$ (1)

or equivalently, using $t$ to represent the departure time,

$$C_{rs}(t) = p_i(t + \tau_{rs}^O) + \alpha(\tau_{rs}^i + f_i(k_i(t + \tau_{rs}^O)))$$ (2)

where $\alpha$ is the average value of time for the traveler population. $f_i(t)$ denotes the vehicle cruising time at parking lot $i$ for a traveler arriving at time $t$. In reality, the cruising time is dependent on many random factors, such as the locations of vacant spaces, cruising routes, etc. Therefore, it is more appropriate to set the individual cruising time as a random variable, while $f_i(t)$ here is the expected average vehicle cruising time. This time is in fact closely related to the parking occupancy and the type of parking information provided to the users. We will discuss more about the expected cruising time in a moment (later we use the phase “cruising time” or “searching time” directly for simplicity purposes).

2.4. Parking searching time

Very few theoretical parking models consider parking cruising time, but it is a key component in parking management and design. In this paper, we assume a general family of searching time functions, $f_i(k_i(t))$, where $k_i(t)$ is the “effective” occupancy at the arrival time $t$, defined by the number of vehicles arriving prior to $t$, i.e., the number of occupied spaces (also know as nominal occupancy) plus the number of vehicles
arriving before \( t \) and cruising in lot \( i \). The reason we adopt the effective occupancy rather than the nominal occupancy is because those arriving prior to \( t \) and cruising usually find spots earlier than those arriving at \( t \), as a result of FIFO. The number of available spaces in the lot for a vehicle arriving at \( t \) to choose from is actually less than the vacant spots observed at \( t \). Therefore, we will use the effective occupancy in the rest of the paper.

The searching time is dependent on the dynamic parking occupancy and the type of parking information provided to the travelers. Generally, the more parking information provided, the less cruising time travelers are subject to. Now consider the case where travelers are provided with the number of vacant spaces in each parking lot with, however, no exact spot location information (this is the most common case in the current parking management system). A typical searching time function for travelers provided with overall occupancy information is shown in Figure 3 (also reported by Axhausen et al. (1994) and Horni et al. (2012)). The function is convex. It is rather flat under low or medium occupancy where parking time is approximately the same as the time spent in an empty lot, \( \epsilon_i \). However, the searching time increases dramatically when the occupancy is high. Especially for the very last few spots, the searching time could be very high. The cruising time is infinite when the lot is fully occupied.

![Fig. 3. A typical parking searching time function for travelers provided with occupancy information](image)

Such expected searching time function may only exist when all travelers using this lot have full access to the availability information (by online information system or day-to-day experience). When such information is not available, the cruising time may be rather random. Travelers may randomly choose one lot, cruise around a bit before he finds no vacancy and proceed to another lot (although there are vacant spots in the first one and they do not find them!). In fact, the real-time availability information ensures a stabilized parking choice in real time.

Thereafter we assume the parking searching time function is strictly monotone with respect to the occupancy and is convex.

### 2.5. Notations

In addition, we use the following convention throughout the paper. Bolded letter is used to represent a vector of real number. For instance, \( \mathbf{x} \) is defined as a list of real number \( x_{i,j} \), where each element has subscripts \( i \) and \( j \). This reads, \( \mathbf{x} = \{x_{i,j}\}_{i,j} \). In the vector, the elements \( x_{i,j} \) are ordered in \( i \) first (for the same \( j \)), and then by \( j \). The same convention also applies to other variables with more subscripts or superscripts.

The following notations are used throughout the paper. For the four indices, \( i, r, s, t \), whenever we use \( \forall \), it means the index is chosen from the full set of that index.

- \( i \) or \( j \), index of a parking lot/block. \( i, j \in \{1, 2, \ldots, n\} \)
- \( r \), index of an origin node. \( r \in R \)
- \( s \), index of a destination node. \( s \in S \)
t, index of a departure time interval. \( t \in \{1, 2, \ldots, T\} \). \( t \) may also be used to represent the index of an arrival time interval to a lot. In this case, \( t \in \{1, 2, \ldots, T, T + 1, \ldots, T + \max_{s,i}(t_0^i)\} \). It takes a traveler who departs home at time \( T \) up to \( \max_{s,i}(t_0^i) \) amount of time period to arrive his targeted parking lot, which extends the time horizon. The former case is assumed in the text if not mentioned otherwise.

The real-time occupancy information (via online information provision or day-to-day experience) helps travelers choose the lot that yields the lowest travel cost for him in real time, which ensures the stabilized travel experience, travelers are familiar with the expected cruising time of those parking lots, and the UE is clearly a day-to-day pattern. The UE defined here is likely to occur in two situations. One is that after day-to-day experience, travelers are aware of three types of information at any time, the real-time occupancy, the expected cruising time, and the real-time parking prices. Given the location, capacity and time-varying prices of all parking lots, the eventual flow patterns are such that, 1) the generalized travel cost of all travelers are the same; 2) No traveler can unilaterally change his parking choice and/or departure time choices to reduce his generalized travel cost.

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Mathematically, the UE condition reads in the following form. Given a feasible parking flow pattern \(|A| \geq 0| \sum \lambda_{rs}^i(t) = \lambda_{rs}(t), \forall r, s, t,\)

\[
\begin{align*}
\lambda_{rs}^i(t) > 0 & \Rightarrow C_{rs}^i(t) = \pi_{rs}(t), \forall r, s, i, t \\
\lambda_{rs}^i(t) = 0 & \Rightarrow C_{rs}^i(t) \geq \pi_{rs}(t), \forall r, s, i, t
\end{align*}
\]

The mathematical formulation states that for any travelers departing at time \(t\) between O-D pair \(rs\), if choosing lot/block \(i\), then his travel cost \(C_{rs}^i(t)\) must attain the minimum of the cost of all the travelers departing at time \(t\) between O-D pair \(rs\), denoted by \(\pi_{rs}(t)\). If none of those travelers choose to park in lot/block \(i\), then the corresponding travel cost of using lot/block \(i\) is greater or equal to the minimum travel cost \(\pi_{rs}(t)\).

The UE condition 5 can be cast into a variational inequality (VI) problem. Define \(C\) as the vector of the travel cost of all the travelers, i.e., \(\{C_{rs}^i(t)\}_{i,r,s,t}\).

**Proposition 3.1.** User Equilibrium parking flow pattern \(\Lambda^*\) is a solution of VI(\(\Omega, C\)), i.e. find \(\Lambda^* \in \Omega\) such that

\[
(A - \Lambda^*)^T C(\Lambda^*) \geq 0 \forall \Lambda \in \Omega
\]

where \(\Omega = \{|A| \geq 0| \sum \lambda_{rs}^i(t) = \lambda_{rs}(t), \forall r, s, t\}\) is the feasible solution set.

**Proof.** Let us now prove that solving VI(\(\Omega, C\)) is equivalent to the UE condition 5.

Suppose \(\Lambda^*\) is a solution of VI(\(\Omega, C\)) and it does not satisfy condition 5. Then there exists \((r', s', i', t')\) such that \(\lambda_{rs'}^{i'}(t') > 0\) and \(C_{rs'}^{i'}(t') = \pi_{rs'}(t')\). We then construct \(\bar{\Lambda}\) in such a way that all the elements equal to those of \(\Lambda^*\) except for \(\lambda_{rs'}^{i'}(t') = \lambda_{rs'}^{i'}(t') - \epsilon\) (\(\epsilon\) is a small positive number) and \(\lambda_{rs'}^{i''}(t') = \lambda_{rs'}^{i''}(t') + \epsilon\), where \(i''\) is where \(C_{rs'}^{i''}(t')\) attains the minimum travel cost \(\pi_{rs'}(t')\). It is then clear that \(\bar{\Lambda} \in \Omega\), but

\[
(\bar{\Lambda} - \Lambda^*)^T C(\Lambda^*) = \epsilon(\pi_{rs'}(t') - C_{rs'}^{i''}(t')) < 0
\]

Therefore, by contradiction, if \(\Lambda^*\) is a solution of VI(\(\Omega, C\)), then it must satisfy condition 5.

On the other hand, let \(\Lambda^*\) satisfy condition 5. Then for \(\forall \Lambda \in \Omega\),

\[
\begin{align*}
\lambda_{rs}^i(t)(C_{rs}^i(t) - \pi_{rs}(t)) &= 0, \forall r, s, t, i \\
\lambda_{rs}^i(t)(C_{rs}^i(t) - \pi_{rs}(t)) &\geq 0, \forall r, s, t, i
\end{align*}
\]

Therefore, \(\forall \bar{\Lambda} \in \Omega\),

\[
(\bar{\lambda}_{rs}^i(t) - \lambda_{rs}^i(t))(C_{rs}^i(t) - \pi_{rs}(t)) \geq 0, \forall r, s, t, i
\]

which is exactly the definition of VI(\(\Omega, C\)).

**Proposition 3.2.** If the searching time function \(f_i\) is strictly monotone for any lot \(i\), then User Equilibrium parking flow pattern \(\Lambda^*\) exists and is unique.

**Proof.** We first consider the generalized travel cost function. \(C_{rs}^i(t)\) is a function of up-to-date occupancy \(k_i(t)\) as indicated by Equation 1, which is furthermore determined in Equation 3 by the parking flow pattern. By the assumption that the searching time \(f_i\) is continuous and strictly monotone with respect to \(k_i\), we conclude that \(C\) is continuous on \(\Lambda\) and is strictly monotone with respect to \(\Lambda\).

The existence and uniqueness of VI(\(\Omega, C\)) is then followed by the fact that \(\Omega\) is nonempty, closed, bounded and convex, and \(C\) is continuous and strictly monotone (Konnov, 2007).

Several algorithms have been developed to solve for VI(\(\Omega, C\)) (e.g., Nagurney, 1993; Konnov, 2007). One of the most widely used is the projection-based algorithm. We first introduce the regularized merit function (Konnov, 2007),

\[
\phi_{\theta}(x) = \max_{\Lambda \in \Omega} [(x - \Lambda)^T C(\Lambda) - \frac{||A - x||^2}{2\theta}]
\]

**Proposition 3.3.** \(\Lambda^*\) is a solution to VI(\(\Omega, C\)) if and only if \(\Lambda^*\) solves \(\max_{x \in \Omega} \phi_{\theta}(x)\).
Proof. Since we have shown that $C$ is continuously differentiable and strictly monotone, the proposition directly follows Konnov (2007) (p. 173, Proposition 13.1).

$\text{Proj}_\Omega(\Lambda - \delta C(\Lambda)) - \Lambda$ (for $\forall \delta > 0$) is a strict ascent direction for $\phi_0(\Lambda)$ whenever $\Lambda$ is not a solution to $\text{VI}(\Omega, C)$. It also measures the optimality condition, i.e. it converges to zero as the projection goes on. Furthermore, we can use the following gap function to examine its convergence to a true UE solution,

$$\sum_{i=1}^n \sum_r \sum_s \lambda_{irs}(t) (C_{irs}(t) - \pi_{irs}(t))$$

The gap function is expected to converge to zero when the UE solution is achieved.

We have formulated the UE parking choices using the variational inequality method, and proposed an efficient projection-based algorithm to solve it with guaranteed convergence. While it does not yield any closed form for the UE parking flow pattern, its existence and uniqueness is proven.

To see the UE parking choices in a more intuitive way, in the beginning of the commuting time when very few spaces in the parking lots are occupied, travelers always choose the lots that are most desirable (closest to their destination or cheapest). As time goes on, the expected parking cruising time increases for each parking lot. The increase may differ considerably among those lots, dependent on the arrival rates to each lot. Travelers may start to switch to other parking lots (rather than the one they preferred in the first place) by making trade-off between the access time (convenience) and the parking searching time (parking congestion). Note that the change in parking searching time for each lot is a result of parking choices of previous travelers.

4. Optimal parking flow and pricing

Under UE behavior assumption, different parking prices lead to different UE flow patterns, and thus different network performance, represented by the total travel cost (including the cruising time and composite travel time). The parking fees collected are generally not considered in the total cost. We are most concerned about the System Optimal (SO) pricing schemes that can lead to the minimal total cost. The resultant UE flow pattern under the SO prices is referred to as the SO flow pattern.

4.1. Formulation

To solve the SO parking pricing, we seek the optimal parking flow pattern that minimizes the total system cost. The optimization problem is,

$$\min_{\Lambda} TC = \alpha \int_0^{\kappa_i} f(x)dx + \alpha \sum_{r \in R} \sum_{s \in S} \sum_{i=1}^n \sum_{t=1}^T \left( t_{irs}^0 + t_{irs}^D \right) \lambda_{irs}(t)$$

subject to:

$$\sum_{i=1}^n \lambda_{irs}(t) = \lambda_{rs}(t), \forall t, r, s$$  \hspace{1cm} (9b)

$$\kappa_i = \sum_{r \in R} \sum_{s \in S} \sum_{t=1}^T \lambda_{irs}(t), \forall i$$

$$\kappa_i \leq K_i, \forall i$$  \hspace{1cm} (9c)

$$\lambda_{irs}(t) \geq 0, \forall r, s, t, i$$  \hspace{1cm} (9d)

where $\kappa_i$ is the number of spaces finally occupied in lot $i$, referred as “terminal occupancy” in the rest of the paper.

The total system cost (TC) is the sum of total cruising time for all the travelers from all the O-D pairs and their total composite travel time. All the parking fees collected by the lots are generally considered as part of the social welfare, and thus are not included in the total system cost. They are either re-distributed to
the public by using those revenue to subsidize the public transit, or are part of the social benefits for parking operators. We want to find the optimal parking flow pattern that minimizes the total cost. Equation 9b states the flow feasibility condition, i.e. the sum of the parking flow with respect to any O-D pair \( rs \) for all lots is equal to the total commuting flow between \( r - s \) at any departure time \( t \), and the flow must be nonnegative in Inequality 9e. Equation 9c computes the terminal occupancy of each lot by aggregating the time-dependent demand over time, and the terminal occupancy is smaller or equal to the parking capacity in Inequality 9d.

We will first solve the optimal terminal parking occupancy \( \{k_i\} \), which is followed by the optimal flow pattern and parking prices in the following subsections.

### 4.2. Optimal terminal parking occupancy

**Proposition 4.2.1.** The optimal terminal parking occupancy \( \{k_i\} \) in the optimization problem 9 can be solved by applying UE static traffic assignment in the general parking network (Figure 2). The static traffic demand is \( \Xi = \{\lambda_{rs}\}_{r,s} \), \( \tau_{ri}^O \), and \( \tau_{ri}^D \) are the constant travel time for links connecting to/from the parking lots. \( f_i(\cdot) \), serve as the variable link performance functions of those parking lots (seeing lots as links in Figure 2).

**Proof.** This proposition can be shown by equalizing the marginal travel cost (MTC) of the entire parking network for all travelers.

The MTC of time \( t \) between O-D pair \( rs \) and using lot \( i \) is the sum of additional system cost caused by sending an additional traveler departing at time \( t \) between O-D pair \( rs \) and using lot \( i \). Such a traveler can increase the total searching time by \( f_i(k_i) \) and its own composite travel time \( \tau_{ri}^O + \tau_{ri}^D \). For any travelers with departure time \( t \) between any O-D pair \( rs \), the MTC of using parking lot \( i \) should be identical, if lot \( i \) is being used. We further observe that MTC consists of the free-flow travel time and the marginal parking searching time, \( f_i(k_i) \), and thus is independent of departure time \( t \). Therefore, we may omit the time dimension, and MTC reads,

\[
\lambda_{ri} > 0 \Rightarrow \frac{\text{MTC}_{rs}}{\alpha} = \tau_{ri}^O + \tau_{ri}^D + f_i(k_i), \forall r, s, i \in L_{rs}
\]

where \( L_{rs} \in \{1, 2, \ldots, n\} \) is the set of parking lots being used by demand between O-D pair \( rs \) during the analysis horizon. On the other hand, if no travelers from O-D pair \( rs \) use lot \( i \), then the corresponding composite travel time and searching time should be greater than the MTC of using any of the lot by the definition of SO.

\[
\lambda_{ri} = 0 \Rightarrow \frac{\text{MTC}_{rs}}{\alpha} < \tau_{ri}^O + \tau_{ri}^D + f_i(k_i), \forall r, s, i \in L_{rs}
\]

The optimality conditions 10 and 11 are essentially equivalent to the UE condition of assigning demand \( \{\lambda_{rs}\}_{r,s} \) to the parking network, by seeing \( \frac{\text{MTC}_{rs}}{\alpha} \) as the minimum cost between O-D pair \( rs \).

Thus, the well-known UE static traffic assignment (STA) problem can be applied here to solve for the optimal terminal parking occupancy. In the UE STA, all the origin connectors (to the lots) and destination connectors (from the lots) have the constant link cost \( \tau_{ri}^O \) and \( \tau_{ri}^D \) respectively, and each of the parking lots are treated as a link with the link performance function \( f_i(\cdot) \).

In addition, Inequality 9d is satisfied when solving UE-STA by choosing appropriate parking searching time. For instance, the searching function \( f_i \) shown in Equations 9d and 9e are such that the searching time of the very last parking space is sufficiently large. Therefore, the terminal occupancy solved by UE-STA will not exceed the capacity if the parking lots can accommodate all the travel demand, i.e. \( \sum_{i=1}^{K_r} K_i > \sum_{rs} \lambda_{rs} \).

Proposition 4.2.1 allows us to solve the optimal terminal parking occupancy efficiently without necessarily working on the optimization problem 9 directly. A conventional method to solve UE-STA is Frank-Wolfe algorithm (LeBlanc et al., 1975). In this case, due to the features of the parking network where only parking lots have variable link performance function and other links are connectors with constant travel time, we can...
Proposition 4.2.2. If the searching time function \( f_i \) is strictly monotone for any lot \( i \), then the optimal terminal parking occupancy \( \{\kappa_i\} \) exists and is unique.

Proof. This proposition is immediately followed by the fact that any traveler must use one of the parking lot and the parking searching time of each lot is strictly monotone. Since the path travel cost in the UES-TA is strictly monotone, the flow, i.e. \( \{\kappa_i\} \), in this case, exists and is unique. Note that the flow on the origin/destination connectors may not be unique. \( \square \)

In fact, we show that the optimal terminal occupancy for each lot is independent of the value of time \( \alpha \). This is essentially because each traveler evaluates the time in the same monetary cost and the direct expenditure (parking fee) is not considered as part of the total cost. In other words, if we assume a homogeneous population, this property allows us to determine the optimal parking usage without necessarily knowing the population value of time.

4.3. Optimal total parking flow

Given the optimal terminal parking occupancy \( \{\kappa_i^*\} \), the optimization problem 9 is then equivalent to the following optimization problem (we let \( d_{rs}^i = c_{rs}^i + \lambda_i^r \)):

\[
\min \sum_{(i,k)} \sum_{r \in R} \sum_{s \in S} \sum_{i=1}^{n} d_{rs}^i x_{rs}^i \tag{12a}
\]

s.t. \( \sum_{r \in R} \sum_{s \in S} x_{rs}^i = \kappa_i^*, \forall i \) \tag{12b}
\[
\sum_{i=1}^{n} x_{rs}^{i} = \sum_{t=1}^{T} \lambda_{rs}(t), \forall r, s, \tag{12c}
\]
\[
\lambda_{rs}^{i} \geq 0, \forall r, s, i \tag{12d}
\]

The optimization problem 12 solves for the optimal total parking flow \( \Theta^{*} = \{\lambda_{rs}^{i}\}_{r,s,i} \) over the entire analysis horizon. Any time-varying parking flow pattern \( \Lambda \geq 0 \) that satisfies the following feasibility conditions is optimal,

\[
\sum_{i=1}^{n} x_{rs}^{i}(t) = \lambda_{rs}(t), \forall r, s, i \tag{13a}
\]
\[
\sum_{i=1}^{n} \lambda_{rs}^{i}(t) = \lambda_{rs}(t), \forall r, s, t \tag{13b}
\]

Let us first focus on solving the total parking flow \( \Theta \). We now write it in matrix notation. Let \( I_{m} \) denote the \( m \)-by-\( m \) identity matrix, and \( O_{m} \) the \( m \)-tuples vector with each element being 1. \( \kappa^{*} \) denotes the vector of the optimal terminal occupancy. Define,

\[
B = \begin{pmatrix} I_{n} & I_{n} & \ldots & I_{n} \\ O_{n} & O_{n} & \ldots & O_{n} \end{pmatrix}_{(n+|R||S|) \times (|R||S|)}
\]

\[
b_{1} = \begin{pmatrix} \kappa^{*} \\ \Xi \end{pmatrix}_{(n+|R||S|) \times 1}
\]

\[
D = \{d_{rs}^{i}\}_{r,s,i}
\]

The optimization problem 12 reads in its matrix notation,

\[
\begin{aligned}
\min_{\Theta} & \quad D \cdot \Theta \\
\text{s.t.} & \quad B \cdot \Theta = b_{1} \\
& \quad \Theta \geq 0
\end{aligned} \tag{14a-b-c}
\]

The linear program 14 is usually under-determined since \( n + |R||S| \ll |R||S| n \). Thus, its optimal solution may not be unique. We here explore its optimal solution set.

**Proposition 4.3.1.** Suppose \( \Theta_{0}^{*} \) is a solution of the linear program 14. Do a singular value decomposition for,

\[
\begin{pmatrix} D^{T} \\ B \end{pmatrix} = U \Sigma V^{T}
\]

where \( U \) is a \( (n + |R||S| + |R||S| n) \times (|R||S| n) \) nonnegative real numbers on the diagonal, and \( \Sigma \) is an diagonal matrix with \( (|R||S| n) \times (|R||S| n) \) real matrix (with orthogonal columns). Let \( m \) be the number of non-zero diagonal elements. If \( |R||S| n - m \leq 0 \), then the linear program 14 has a unique solution \( \Theta_{0}^{*} \). Otherwise, we can choose the last \( |R||S| n - m \) columns of \( V \), denoted by \( \tilde{V} \). Search any \( (|R||S| n - m) \)-tuples vector \( \gamma \) such that,

\[
\tilde{V} \gamma \geq -\Theta_{0}^{*}
\]

and \( \tilde{V} \gamma + \Theta_{0}^{*} \) is also an optimal solution of the linear program 14.

**Proof.** Suppose \( \Theta_{0}^{*} + z \) is also a solution of the linear program 14. Then,

\[
\begin{cases}
D^{T} z = 0 \\
B z = 0 \\
\Theta_{0}^{*} + z \geq 0
\end{cases}
\]
For any $z$ in the basis of $\tilde{V}$, i.e., $z = \tilde{V}_Y$, by the definition of $\tilde{V}$ in the proposition,
\[
\begin{pmatrix}
D_Y^T \\
B
\end{pmatrix}
\begin{pmatrix}
\tilde{V}_Y
\end{pmatrix}
= \begin{pmatrix}
U \Sigma V_T \tilde{V}_Y
\end{pmatrix}
= 0
\]
which completes the proof.

By Proposition 4.3.1, we are able to find any optimal total parking flow $\Theta^*$. It is not unique in most of the cases. This will be further discussed in the next subsection, as we will see the optimal dynamic parking flow pattern is not unique as well.

**4.4. Optimal dynamic parking flow pattern**

Next we can solve for the optimal dynamic parking flow pattern $\Lambda^*$. 

**Proposition 4.4.1.** The optimal parking flow pattern $\Lambda$ that solves the optimization problem 9 exists and is not unique.

**Proof.** It is easy to show that the optimization problem 12 combining with the feasibility condition 13 determines the optimal parking flow pattern that solves the optimization problem 9. For the optimization problem 12, the feasible set determined by Equations 12b, 12c and 12d is nonempty, since $\kappa_i^* = \sum_r \sum_s \lambda_{rs}^t$ and $\lambda_{rs}(t) > 0, \forall t$. Thus, the linear programming of the optimization problem 12 can always attain a minimum with at least one solution $\Theta^*$. Clearly, given any $\Theta^*$, $\Lambda = \{\lambda_{rs}(t)\}_{r,s,t}$ determined by condition 13 is not unique.

Given an optimal total parking flow $\Theta^*$, the feasible condition of the optimal $\Lambda^*$ reads (writing the feasibility condition 13 in matrix notation),
\[
\begin{pmatrix}
A_1 \\
A_2
\end{pmatrix}
\Lambda
= 
\begin{pmatrix}
\Gamma \\
\Theta^*
\end{pmatrix}
\]
\[
\Lambda \geq 0
\]
where $\Gamma$ denotes the vector form of the parking demand $\{\lambda_{rs}(t)\}_{r,s,t}$, and,

\[
A_1 = \begin{pmatrix}
O_n \\
O_n \\
\ddots \\
O_n
\end{pmatrix}_{|R||S| \times |R||S|}
, A_2 = \begin{pmatrix}
I_{|R||S|} & I_{|R||S|} & \cdots & I_{|R||S|}
\end{pmatrix}_{|R||S| \times |R||S|nT}
\]

Since the optimal dynamic parking flow pattern $\Lambda^*$ is not unique, we could further determine an optimal one according to other traffic management objectives. Consequently, we can propose a generic management objective $F(\Lambda)$ to finally choose the most desirable optimal parking flow pattern, subject to feasibility conditions 15.

\[
\begin{align*}
\min_{\Lambda} F(\Lambda) \\
\text{s.t.} \quad & \begin{pmatrix}
A_1 \\
A_2
\end{pmatrix}
\Lambda
= 
\begin{pmatrix}
\Gamma \\
\Theta^*
\end{pmatrix} \\
& \Lambda \geq 0
\end{align*}
\]

Note that some rows in the composite matrix $A_1$ and $A_2$ are redundant, since $\sum_r \lambda_{rs}^t = \sum_s \lambda_{rs}(t), \forall r \in R, s \in S$. In real implementation, we should remove the last $|R||S|$ rows of matrix $A_1$ to ensure every row in the composite matrix ($A_1$ and $A_2$) is linearly independent.

Those objectives $F(\cdot)$ include minimizing roadway congestion, temporal or spacial restriction on parking flow, priority parking and so forth. For example, parking operators may reserve the usage of a particular lot
The time-varying searching time $f_{rs}(t)$ may be concerned about the roadway congestion caused by queuing at the parking entrance and parking cruising (the rationale is discussed in Section 2.2). The roadway congestion is not explicitly considered in the SO formulation, but it could be reduced by carefully choosing appropriate arrival rates $\Phi = [\lambda_i(t)]_{i,t}$. A set of desirable arrival rates could be,

$$
\lambda_i(t) = \frac{\kappa_i^*}{T + \max_j[\tau_{ij}^0]} \forall i, t \in [1, 2, \ldots, T']
$$

Given the terminal parking occupancy, the minimal queue at the parking entrance is achieved by setting a constant arrival rate. In this case, the objective $F(\cdot)$ reads,

$$
\min_{\Lambda} \|A_3\Lambda - \Phi\|^2
$$

where, $A_3$ is a $nT \times |R||S|nT$ matrix. For each row of $A_3$ (i.e. given a pair of $i$ and $t$), the elements corresponding to $\lambda_i^*(t - \tau_{ij}^0)$ is one for all the OD pair $r - s$, and other elements are zeros.

To sum up, we need to solve two optimization problems sequentially, Problem 14 and Problem 16, in order to find the most desirable SO parking flow pattern. Both problems 14 and 16 require large-scale algorithms for general networks. The dimension of the decision variable $\Lambda$ is $|R||S|nT$, but it could be reduced considerably during the solution procedure. If we implement the simplex method for solving the linear program 14, then the solution $\Theta^*$ is usually sparse (i.e., most of the elements are zeros) for large-scale networks. In addition, notice that

$$
\lambda_i^* = 0 \Rightarrow \lambda_i^*(t) = 0, \forall t
$$

For any O-D pair $rs$ and parking lot $i$ such that $\lambda_i^* = 0$, $T$ amount of the elements in $\Lambda$ could be eliminated. The final dimension in $\Lambda$ is much smaller than $|R||S|nT$. Furthermore, large-scale iterative algorithms methods for solving sparse rectangular systems, as shown in Problem 16, have been developed, such as LSQR or LSRM.

### 4.5. Optimum parking pricing schemes

The optimal parking flow pattern $\Lambda^*$ can be realized by charging optimal parking prices. The optimal parking prices are such that the generalized travel costs (inclusive of the parking fee) of any travelers from the same O-D pair and with the same departure time are identical, and meanwhile the parking flow pattern is optimal as derived before. In general, two types of parking prices can be charged in each parking lot, and their definitions are given as below.

**Definition 4.5.1.** A dynamic parking pricing scheme is path-based, if for each lot, the parking fee is charged by travelers’ O-D and their arrival time to the lot.

**Definition 4.5.2.** A dynamic parking pricing scheme is lot-based or link-based, if for each lot, the parking fee is charged by travelers’ arrival time to the lot, regardless of their origins and destinations.

Note that since the parking fees are imposed with respect to travelers’ arrival time, the discretized time interval of parking fee is such that $t \in [1, 2, \ldots, T, T + 1, \ldots, T']$.

Clearly, there always exists a path-based parking pricing scheme to achieve the optimal parking flow pattern. To see this, for $\forall r, s, t, i$, we have,

$$
\lambda_i^*(t) > 0 \Rightarrow d_{rs}^i + f_i(k_i(t + \tau_{ij}^0)) + p_i^r(t + \tau_{ij}^0) = \pi_{rs}(t)
$$

$$
\lambda_i^*(t) = 0 \Rightarrow d_{rs}^i + f_i(k_i(t + \tau_{ij}^0)) + p_i^r(t + \tau_{ij}^0) \geq \pi_{rs}(t)
$$

$$
\pi_{rs}(t) = \arg \min_{m \in \{1, 2, \ldots, n\}} \left[ d_{rs}^i + f_i(k_i(t + \tau_{ij}^0)) + p_i^r(t + \tau_{ij}^0) \right]
$$

The time-varying searching time $f_i(k_i(t + \tau_{ij}^0))$ is determined by the optimal pattern flow pattern $\Lambda^*$. Therefore, solving Equation 19 for each O-D pair $rs$ and departure time $t$ always yields a set of feasible path-based parking prices $\{p_i^r(t)\}_{r,s,t,i}$.
Though the path-based optimal parking pricing is always feasible, it may not be realistic. This is due to the difficulty of tracking the origin or destination of each vehicle upon the arrival. The link-based parking pricing scheme is what most parking lots adopt currently. Being easily implementable and realistic, the link-based parking fees seem to lose the ability of balancing travelers’ parking choices between any O-D pairs, as compared to the path-based fees. A question naturally arises whether such a link-based pricing scheme can also achieve the SO parking flow pattern? Fortunately, the answer is positive by the following proposition.

**Proposition 4.5.1.** Any optimal parking flow pattern $\Lambda^*$ can be realized by a lot-based parking pricing scheme $P = \{ p_i(t) \}_{t,i}$.

Note that the composite travel time matrix $D$ is path-dependent. Before proving 4.5.1, we first show and prove the following lemma.

**Lemma 4.5.1.** Any optimal solution $\Lambda^*$ must satisfy, for $\forall t, t' \in \{1, 2, \ldots, T \}, r, r' \in R, s, s' \in S, i, j \in \{1, 2, \ldots, n\}, i \neq j$,

\[
\begin{align*}
\lambda^{i}_{r's}(t)\lambda^{j}_{r} (t') & = 0 & \text{if } d^{i}_{r's} - d^{j}_{r} < d^{i}_{r's} - d^{j}_{r's} \\
\lambda^{j}_{r}(t)\lambda^{i}_{r's}(t') & = 0 & \text{if } d^{j}_{r} - d^{i}_{r's} > d^{j}_{r's} - d^{j}_{r's}
\end{align*}
\]

(20)

**Proof.** Let us examine the linear program 12. Attach Lagrange multiplier $\rho_i, \mu_{rs}, \nu_{rs}$ to the constraints 12b, 12c and 12d, respectively. The first-order KKT condition reads,

\[
\frac{\partial L}{\partial \lambda^{i}_{rs}} = d^{i}_{rs} + \rho_i + \mu_{rs} - \nu_{rs} = 0, \lambda^{i}_{rs} = 0, \forall r, t, i
\]

First of all, we must have at least one of the four, $\lambda^{i}_{r's}, \lambda^{j}_{r's}, \lambda^{i}_{r's}$, and $\lambda^{j}_{r's}$, to be zero. If all these four terms are non-zero, then $\nu_{r's} = \nu_{r's} = v_{rs} = v_{r's}$ = 0 by the complementary slackness condition. This results in $d^{i}_{rs} - d^{j}_{rs} = d^{i}_{r's} - d^{j}_{r's}$ from the KKT condition.

Without loss of generality, let $\lambda^{i}_{rs} = 0$, then $\lambda^{i}_{rs}(t) = 0, \forall t$ and $\nu_{rs} > 0$. If one of the two, $\lambda^{i}_{r's}$ or $\lambda^{j}_{r's}$ is zero, then condition 20 holds. Thus, let us consider the case where both $\lambda^{i}_{r's}$ and $\lambda^{j}_{r's}$ are non-zero. Then, $\nu_{r's} = 0, v_{rs} = 0$. We shall have $d^{i}_{rs} - d^{j}_{rs} = d^{i}_{r's} - d^{j}_{r's} + v_{r's}$. Hence, $d^{i}_{rs} - d^{j}_{rs} > d^{i}_{r's} - d^{j}_{r's}$. Similarly, we can also show that if $\lambda^{j}_{r's} = 0$ or $\lambda^{i}_{r's} = 0$, then $d^{i}_{rs} - d^{j}_{rs} < d^{i}_{r's} - d^{j}_{r's}$. \( \blacksquare \)

Now we provide the proof of Proposition 4.5.1.

**Proof.** The lot-based parking pricing scheme $\{ p_i(t) \}_{t,i}$ must satisfy that, for $\forall i, j \in \{1, 2, \ldots, n\}, i \neq j, \forall r, s, t$

\[
\begin{align*}
\lambda^{i}_{r}(t) > 0, d^{i}_{rs} + f_i(k_i(t + \tau^{O}_{i,j})) + \frac{p_i(t + \tau^{O}_{i,j})}{\alpha} & = d^{i}_{rs} + f_j(k_j(t + \tau^{O}_{i,j})) + \frac{p_j(t + \tau^{O}_{i,j})}{\alpha} \\
\lambda^{j}_{r}(t) & \geq 0
\end{align*}
\]

(21a)

\[
\begin{align*}
\lambda^{i}_{r}(t) > 0, d^{i}_{rs} + f_i(k_i(t + \tau^{O}_{i,j})) + \frac{p_i(t + \tau^{O}_{i,j})}{\alpha} & < d^{i}_{rs} + f_j(k_j(t + \tau^{O}_{i,j})) + \frac{p_j(t + \tau^{O}_{i,j})}{\alpha} \\
\lambda^{j}_{r}(t) & = 0
\end{align*}
\]

(21b)

Notice that the searching time $f_i$ and price $p_i$ are lot-based, while the composite travel time $d$ is path-based. Therefore, the lot-based parking prices are constrained as follows. For any $\forall t, t', r, r', s, s', i, j$ such that $i \neq j, t + \tau^{O}_{i,j} = t + \tau^{O}_{r,s}$ and $t + \tau^{O}_{r,s} = t + \tau^{O}_{r,s'}$, we have (use index $r, s, t$ for condition 21a and index $r', s', t'$ for condition 21b),

\[
d^{i}_{rs} - d^{i}_{rs'} < d^{i}_{r's} - d^{i}_{r's'} \Rightarrow \lambda^{i}_{rs}(t)\lambda^{j}_{r's}(t') = 0
\]

(22)

Similarly, if we use index $r', s', t$ for condition 21a and index $r, s, t'$ for condition 21b, then we have,
\[ d^l_{rs} - d^l_{rs} > d^l_{rs} - d^l_{rs} \Rightarrow \lambda^l_{rs}(t)\lambda^l_{rs}(t') = 0 \]  

Clearly, the required conditions 22 and 23 are guaranteed by the linear program 12 as a result of Lemma 4.5.1.

Lemma 4.5.1 shows that the condition 20 should be added to the optimization problem 16 as a new constrain. Only if this additional constraint is enforced, the resultant optimal parking flow pattern can be achieved by lot-based parking pricing schemes. Recall that we need to solve two optimization problems sequentially, Problem 14 and Problem 16, for an SO parking flow pattern. We show that by Proposition 4.5.1, this additional constrain should be automatically guaranteed when we solve the linear program 14 (i.e. the matrix form of the linear program 12).

The conditions 21a and 21b allow us to solve for the optimal lot-based parking prices.

For travelers with departure time \( \forall t \) and parking choice \( \forall i, j, i < j \), the required conditions 22 and 23 are guaranteed by the linear program 12 as a result of Lemma 4.5.1.

4.5.1. This additional constrain should be automatically guaranteed when we solve the linear program 14 (i.e. the matrix form of the linear program 12).

We show that by Proposition 4.5.1, this additional constrain should be automatically guaranteed when we solve the linear program 14 (i.e. the matrix form of the linear program 12).

The conditions 21a and 21b allow us to solve for the optimal lot-based parking prices.

For travelers with departure time \( \forall t \) and parking choice \( \forall i, j, i < j \), the required conditions 22 and 23 are guaranteed by the linear program 12 as a result of Lemma 4.5.1.

Clearly, conditions 27 constitute an under-determined system. While any parking prices satisfying the conditions 27 are optimal, other criteria (denoted by function \( G(\cdot) \)) on selecting the prices should be imposed to make it realistic. For instance, we set,

\[ G(\mathbf{P}) = ||E\mathbf{P} - \epsilon||^2 \]
Let $E = I_{nT}$ and $e = 0$, we are minimizing the norm of the price vector, namely the overall SO lot-based price values are set to be minimal. If we set $E = I_{nT}$ and $e$ monotonically increasing, then we expect the SO lot-based prices to approach the pattern of linear increasing during the peak time. We may also let the SO prices approach a desired overall average price $p$ by setting $E = I_{nT}$ and $e = pO_{nT}$.

In general, we solve the following optimization problem to obtain the optimal lot-based parking prices,

$$
\min \ G(P) \quad \text{s.t. Conditions 27(a), (b) and (c)}
$$

where conditions 27(a) and (b) are built on the optimal parking flow pattern $\Lambda^*$.

4.6. Discussions on the optimal solution

We further explore some interesting results accompanying the SO formulas.

If for $i, j \in \{1, 2, \ldots, n\}, i \neq j, r, s, t,$ and travelers between O-D pair $rs$ and departing at time $t$ choose both lots $i$ and $j$, then,

$$
a d_{ij} + \alpha f_i (k_i (t + \tau_{ij}^O)) + p_i (t + \tau_{ij}^O) = a d_{ij} + \alpha f_j (k_j (t + \tau_{ij}^O)) + p_j (t + \tau_{ij}^O)
$$

As we can see, travelers departing at time $t$ make parking choices based on the real-time occupancy and prices up to his arrival time $t + \tau_{ij}^O$. Provision of parking information plays an important role in determining the parking choices, and thus the SO flow pattern.

Differentiate both sides with respect to $t$, we obtain (where $*$ denotes the optimal solution),

$$
p_i^* (t + \tau_{ij}^O) - p_j^* (t + \tau_{ij}^O - 1) + \alpha \frac{\partial f_i (k_i^* (t + \tau_{ij}^O))}{\partial k_i} \lambda_i^* (t)
$$

Equation 32 describes the relations between optimal parking fee and the real-time occupancy. Suppose no traveler is assigned to use a lot by the optimal pattern currently (i.e., $\lambda_i^* (t) = 0$), but it will be used later on. Then its optimal price change is equal to the marginal searching time of another lot being used during this time period, and this marginal searching time is equalized among all the lots currently being used. During this time interval, a lot with less occupancy is assigned with more travelers.

Provided that a lot is used during a time period, the change in the optimal price (lot-based) equals the difference of the marginal searching time between another lot being used and that lot, and thus the optimal price change is negatively related to its own real-time occupancy (assuming the searching time function is convex, as discussed in Section 2.4). In other words, if a lot is used during a time period in its optimal flow allocation, then the change in the optimal parking price should go down as the occupancy increases (while the price still goes up). This is because now providing occupancy information to travelers can also effect travelers’ parking choice, setting the price too high may otherwise prevent the facility being fully used.

Equation 32 indicates that the parking price and the provision of parking information jointly serve as effective ways of managing traffic. The parking price alone may not be as much efficient, because the cruising time, as a key component of the generalized travel cost, is not considered and cannot be affected by the price alone. While the real-time occupancy (or the estimated cruising time) conveys the information of parking congestion level, it also effectively adjusts travelers’ parking choices in real time. With information provision, parking price is not the only economic manner to manage the traffic. The parking price and the information provision should be balance out and work jointly for the best system performance.

Lemma 4.5.1 shows that any two parking lots cannot be used by travelers between more than one O-D pairs at any time, unless the composite travel time follows a special (and rare) relationship. In most of the
cases where \( d_{i,r,s} - d_{j,r,s'} \neq d_{i,r,s'} - d_{j,r,s} \) for \( \forall r, r', s, s', i, j \), it is very unlikely that travelers from different O-D pairs share the more than one parking lots at the same time. The only exception is that their perceived travel time difference between the most preferred two lots are approximately the same. This difference can be understood as the measurement of their preference to the two lots based on their roadway travel time. However, it is possible that travelers with the same O-D choose multiple parking lots at the same time. They make tradeoff between the parking preference, parking prices and the estimated parking congestion.

The results described in Proposition 4.5.1 are appealing. Any optimal parking flow pattern can be achieved by a lot-based parking pricing scheme. This nice feature makes the SO parking pricing scheme easy to implement. Because the parking price and the occupancy is a one-to-one mapping, each lot can set the the optimal real-time parking pricing based on the real-time occupancy, without necessarily tracking the origins and destinations of parking demand. This provides a strong support that optimal parking prices and information provision are able to effectively manage the traffic. When the roadway congestion on the arterials are relatively small compared to the parking cruising time, the lot-based time-varying parking prices are fully capable of adjusting travelers’ route choices and parking choices in its optimal form. However, the dynamic prices set on the lots may not be able to fully eliminate heavy roadway congestion, and this deserves further investigation in future research.

5. Numerical experiments

In this section, we provide two numerical experiments to quantitatively examine the effectiveness of the optimal lot-based parking prices. We shall compare the parking patterns obtained from both UE and SO, with their corresponding system performance. We adopt the searching time function proposed and calibrated by Axhausen et al. (1994) and Horni et al. (2012).

We first choose a two-origin-two-destination synthetic parking network. The small network is used to particularly explain the relations between parking prices, occupancy and the optimal pattern. This can provide additional insights besides the mathematical derivation. The second example is a general parking network of Stanford University campus. We use this example to show the optimal parking flow, and its focus is to provide optimal parking pricing scheme in real networks and illustrate its improvement in network performance compared to UE.

5.1. A synthetic two-origin-two-destination parking network

A synthetic two-origin-two-destination parking network is shown in Figure 4. Nodes 3 and 4 are the origin nodes, and nodes 5 and 6 are the destination nodes. Travelers are between four O-D pairs in this parking network, and they choose from the two parking lots, Lots 1 and 2. The driving time and walking time are marked in the figure (in the unit of seconds). We suppose node 5 is a central business district (CBD), which attracts 80% of the demand from nodes 3 and 4, while the rest heads for the other destination, node 6, a farther office area. Lot 1 is close to the CBD area with 800 spaces, and Lot 2 is close to the node 6 with 800 spaces as well. Clearly, Lot 1 is more preferable by demand from node 3 (also most of the travelers) than Lot 2, while Lot 2 is preferred by demand from node 4. To simulate the traffic demand in the rush hours, we synthesize a trapezoidal-shaped demand for both nodes 3 and 4. Assuming both origins nodes are residential areas with the same size, their demands are the same, i.e., 50, 100, 150, 150, 100, 50 in six 45-min time periods. Therefore, 960 travelers who head for node 5 prefer Lot 2, and only 240 travelers who head for node 6 prefer Lot 2, if the parking fees charged on both lots are equal. The value of time \( \alpha \) is one cent per second, and the parameter \( \epsilon \) for the searching time function 3 is 30 seconds.

We first assume the parking fees charged on both lots are constant for the UE case. The parking fee of Lot 2 is set to $2. Without loss of generality, the fee of Lot 2 changes from $2.1 to $10, just to examine the effect of the parking price. The convergence criteria is set to the value of the gap function 8 is smaller than 0.001, or the maximum 1,000 iteration (for the projection-based algorithm), whichever comes first. The results of UE with respect to the constant parking fee of Lot 1 under UE are shown in Figure 5.

As the parking fee of Lot 1 increases from $2.1 to $10, the composite travel time cost first stays approximately constant around $8400 until the fee of $6, and further on increases significantly to $9600, by
Fig. 4. A synthetic two-origin-two-destination parking network

Fig. 5. The results of UE with respect to the constant parking fee of Lot 1

14%, when the price achieves $10. The total cruising time cost is at a smaller magnitude compared to the composite travel time cost, it attains its minimal when the price of Lot 1 is set to around $6.1. The total travel cost almost follows the same profile as the composite travel time cost. When the parking fee of Lot 1 is still low enough to attract most of the travelers to park in it, the total travel cost is nearly stable and slightly decreasing as the price goes up, thanks to the reduction of cruising time cost. In this range, increasing the parking fee of the central area can effectively reduce the cruising time and thus the total travel cost. However, when the price is set to beyond $6.5, all travelers would prefer to park in Lot 2. In this case, the parking demand to Lot 2 is so intensive that it creates additional cruising time cost, while its total travel time cost could also increase due to the inconvenience of using Lot 2. This is also reflected by the changes in terminal occupancy in both lots. The terminal occupancy reduces from 0.9 to 0.6 for Lot 1 as the Lot 1 price goes up, and meanwhile the terminal occupancy in Lot 2 increases from 0.65 to 0.9. Overall, if we charge a constant parking fee in the closer Lot, then $6.1 is the best. The resultant cruising time cost, composite travel time cost and total travel time cost are $8702, $1997, and $10699. The reason it works best in terms of the system cost is because it balances the demand between Lots 1 and 2. Their respective terminal occupancies (around 0.85 and 0.7 for Lots 1 and 2, respectively) are medium for the entire range of Lot 1 prices. At this price, the cruising time is minimal, while travelers can still take advantage of the convenience of Lot 1 (spaces are used by 85%).

Later on, we choose the constant price $4 and $8 as typical UE cases to compare with the SO solution. The former (UE case 1) represents the case where the price on the preferred lot is relatively low and the preferred lot is in short of supply, while the latter (UE case 2) is a typical case with over-priced fee on the preferred lot and the it is not efficiently used.

We now solve the optimal terminal occupancies, SO total parking flow, SO parking flow pattern and SO prices sequentially. Under SO, the best terminal occupancy for Lots 1 and 2 is 87.5% and 62.5% (namely
700 spots and 500 spots in total), respectively. Although the SO terminal occupancies are fairly close to the UE case 1 where Lot 1 is priced at $4, the UE case 1 is far away from being optimal in total travel cost, as well as its flow pattern. The SO total parking flow are that of the two UE cases are summarized as in Table 1.

<table>
<thead>
<tr>
<th>Solution type</th>
<th>Lot 1</th>
<th>Lot 2</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>3-5</td>
<td>4-5</td>
</tr>
<tr>
<td>SO</td>
<td>220</td>
<td>480</td>
</tr>
<tr>
<td>UE $p_2 = $4</td>
<td>307</td>
<td>398</td>
</tr>
<tr>
<td>UE $p_2 = $8</td>
<td>88</td>
<td>480</td>
</tr>
</tbody>
</table>

In all those cases, the travelers heading for node 6 (outside of CBD) always choose the farther lot (Lot 2) as it is more convenient and less costly for them. When Lot 1 is under-priced (over-used), some travelers from node 4 heading for the CBD may use the less convenient lot (Lot 2). Meanwhile, travelers from node 3 heading for the CBD are more likely to choose the preferred lot than in the SO pattern, though it is subject to greater parking congestion than being optimal. In contrast, if Lot 1 is over-priced (under-used), it is those travelers departing from node 3 and heading for the CBD who are affected most compared to the SO. They are more likely to choose the farther lot rather than the preferred lot (Lot 1). After all, for travelers heading for CBD, the optimal usage of parking spaces are such that all those from node 4 use the preferred lot, as it is more preferable than Lot 1 is to travelers from node 3. Under SO, 46% travelers from node 3 use the preferred lot, and the rest 54% use the less convenient one.

The parking flow patterns obtained from SO and the two UE cases are plot in Figure 6, where the time axle is with respect to the arrival time to the lot. SO keeps up approximately the same terminal occupancy as the UE case 1 where the preferred lot is under-priced, but its arrival rates are milder and close to a constant arrival rate over time. This indicates that under SO, the preferred lot is priced relatively high compared to the less convenient lot in the first place, but not to the level as in the UE case 2. When the parking demand starts to drop, it is then priced relatively low to attract travelers and to ensure it is used efficiently fully up to the optimal terminal occupancy. This conclusion is consistent with Arnott and Rowse (1999); Qian et al. (2012). In those papers, it is suggested that inward parking preference (parking fee is priced such that the farther lot is first preferred and the closer lot is preferred in later stages of the commuting time) yields system optimal. Overall, the SO price should be set properly to produce the terminal occupancy with certain
amount of available spots in the preferred lot (12.5% in this case). Such an SO terminal occupancy should be less than the one with under-priced parking fee in the preferred lot (UE case 2), which also supports Shoup’s parking pricing theory (Shoup, 2005).

Last but not least, the composite travel time cost, cruising time cost and total travel cost resulted from the SO pattern is $8,232, $2,203 and $10,435, respectively. The total travel cost is a 3% reduction of the best constant parking price on Lot 1 (where the three numbers are $8,702, $1,997 and $10,699, respectively). Comparing the two cases, the SO solution increases the cruising time slightly, while it yields the minimal composite travel time. Overall, SO leads to the best system performance. Note that the percentage reduction of total travel cost is on the basis of the composite travel time. The composite travel time here is a relative number among travelers (omitting their travel time on the freeway). For instance, it can be subtracted or inflated by the same amount of time. Thus, the reduction percentage can vary considerably as we assign different composite travel time (while the difference remains the same), but still yield the same flow pattern.

The time-varying SO parking prices and the resultant cumulative occupancy are shown in Figure 7. The criteria $G(\cdot)$ (Equation 29) we use to select the SO price from the SO solution set is $E = I_{nt}$ and $e = 5$, namely, we want to set the base price in Lot 1 to $5. In addition, we also add an additional constrain that $p_i(t)$ is never decreasing over time. We see that the preferred lot is then always priced at $4.8$ over time (except the first few minutes), while the optimal price of the farther lot (Lot 2) starts at 0 (free) in the beginning. As time goes on, the price gradually increases in Lot 2, and it grows faster. The SO prices are such that it favors the farther one by under-pricing the preferred lot in the first place. The farther lot should then be price higher to encourage travelers to use the closer lot at the later stages of the commuting time. Finally, both lots are priced approximately the same at $4.8$. That is exactly the reason why the SO price of Lot 2 increases dramatically once its occupancy hits 50%. The SO prices for Lot 2 is an increasing function (convex) of its occupancy. The optimal price-occupancy diagram, as shown in Figure 7(b), can be used for both lots to set the real-time prices, solely based on the real-time occupancy.

5.2. A parking network of Stanford University campus

Now we choose a real parking network on the campus of Stanford University. There are 7 major parking lots located on the western campus, S-1, S-2, L-17, L-21, L-27, L-29 and L-18/20. The locations of the parking lots and the traffic analysis zones (A,B,C,D) are shown in Figure 8. The capacities of the lots are shown in Table 3. Of all the lots, S-1 and S-2 are the closest to the central campus, and is generally preferred by all travelers (also most congested during a regular weekday currently). The four zones represent the four

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\textsuperscript{1}This constrain may not hold for other parking networks.
destinations where the majority of the traffic demand go from the western entrance of the campus. We rarely observe any roadway congestion on campus, so our model would fit well for this parking network.

Currently, all lots are available for commuters with certain parking permits. There are a small fraction of spaces equipped with meters that are designated for visitors. The parking fee of a lot we use for this experiment is the average daily cost of the permits allowed for that lot. The fees are $6, $4, $2, $4, $2, $2, $2, $2, respectively. We also collected 15-min traffic count data from the four major inbound arterials, northern entrance, northwestern entrance, western entrance, and southwestern entrance. The four inbound arterials are considered as the four origin nodes in this parking network. The driving time and walking time is estimated approximately based on the distance measured in Google Maps. In addition, the time horizon for this analysis is 7:00am-10:00am.

![Fig. 8. A parking network of Stanford University campus](image)

The traffic demand of the four entrances of the campus is shown in Table 2. As can be seen, the traffic increases quickly after 7:00am and almost levels off from 7:30am -9:30am. Finally, it reduces a bit at 10:00am. We further assume the proportions of travelers heading for each of the four campus zones (destinations) are 4:2:4:1. The proportions are inferred based on a rough estimation of the office/classroom densities.

Other parameters of the network are: The value of time \( \alpha \) is set to $35 per hour (approximately the average hourly pay rate on campus); \( \varepsilon \) for the searching time function 3 is set to 30 seconds; The convergence criteria for UE is set to the value of the gap function 8 is smaller than 0.005. In addition, the criteria \( G(\cdot) \) (Equation 29) we use to select the SO price from the SO solution set is \( E = I_{OT} \) and \( e = 4O_{OT} \) (i.e. the expected average parking price is $4 for the entire network).

<table>
<thead>
<tr>
<th>Period (15min)</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
<th>11</th>
<th>12</th>
</tr>
</thead>
<tbody>
<tr>
<td>Southern</td>
<td>120</td>
<td>154</td>
<td>144</td>
<td>183</td>
<td>161</td>
<td>172</td>
<td>156</td>
<td>142</td>
<td>158</td>
<td>149</td>
<td>121</td>
<td>102</td>
</tr>
<tr>
<td>Western</td>
<td>64</td>
<td>92</td>
<td>138</td>
<td>116</td>
<td>84</td>
<td>98</td>
<td>117</td>
<td>97</td>
<td>106</td>
<td>99</td>
<td>75</td>
<td>48</td>
</tr>
<tr>
<td>Northwestern</td>
<td>43</td>
<td>67</td>
<td>46</td>
<td>49</td>
<td>41</td>
<td>63</td>
<td>56</td>
<td>73</td>
<td>62</td>
<td>61</td>
<td>61</td>
<td>40</td>
</tr>
<tr>
<td>Northern</td>
<td>79</td>
<td>91</td>
<td>126</td>
<td>120</td>
<td>102</td>
<td>132</td>
<td>116</td>
<td>116</td>
<td>93</td>
<td>114</td>
<td>100</td>
<td>74</td>
</tr>
</tbody>
</table>

We solved both UE (using aforementioned pre-determined parking prices) and SO on this parking network. The terminal occupancies are shown in Table 3. Compared to the UE (which is close to the currently parking flow pattern), SO assigns less travelers for all lots except the Lot L-18/20 and S-1. Since both S-1 and S-1 are preferred by most travelers for their superior convenience, they should be used nearly 95%
under SO as a balance of convenience and great cruising time. This number is very close to the UE case (also the current case). The SO solution would direct travelers using all the other lots (except S-1) to Lot L-18/20 in the optimal flow pattern. This reduces the occupancies of those lots by 1%-12% (except minor 0.6% increase for S-1), and ensures more efficient usage of Lot L-18/20 with a terminal occupancy 57%, an increase by 11%. Lot L-18/20 is the least convenient lot overall, but it is currently a bit over-priced, at the same price as Lot 17, L-27 and L-29. Under SO, it should be used by more travelers, and those shifting to it can benefit from less cruising time and paying much smaller fee.

<table>
<thead>
<tr>
<th>Table 3. The terminal occupancies of the seven parking lots</th>
</tr>
</thead>
<tbody>
<tr>
<td>S-1</td>
</tr>
<tr>
<td>------</td>
</tr>
<tr>
<td>Capacity</td>
</tr>
<tr>
<td>Occ of UE Spots</td>
</tr>
<tr>
<td>Percentage</td>
</tr>
<tr>
<td>Occ of SO Spots</td>
</tr>
<tr>
<td>Percentage</td>
</tr>
</tbody>
</table>

The optimal total parking flow is shown in Table 4. We can easily verify that there do not exist any two O-D pairs that use more than one parking lot during the entire time horizon.

<table>
<thead>
<tr>
<th>Table 4. The optimal total parking flow: a list of O-D pairs with positive parking flow</th>
</tr>
</thead>
<tbody>
<tr>
<td>S-1</td>
</tr>
<tr>
<td>1, 2, 3, 4 → B</td>
</tr>
<tr>
<td>1 → A</td>
</tr>
<tr>
<td>4 → D</td>
</tr>
</tbody>
</table>

The optimal parking flow pattern is shown in Figure 9. The real-time occupancy for all the seven lots, as a result of the SO pricing, should increase almost linearly as time goes on and finally achieves the optimal terminal occupancy. The flow patterns of Lot S-1 and L-18/20 (the two garages with greatest changes) under both SO and UE are shown in Figure 10. Compared to UE, Lot S-1 should be priced in the way that it is used more mildly between 8:00am to 9:00, and used more intensively afterword. Therefore, we should set a relatively higher price between 8:00am to 9:00am then the UE case, and lower price after 9:00am. Under UE, Lot L-18/20, the least convenient one, is not used til 8:45am. However, the SO pricing ensures it is used from the beginning of the commuting time with a milder arrival rate overall.

A possible set of optimal parking prices and its relation to the real-time occupancy are shown in Figure 11. Between 7:00am and 8:30am, the prices are set to $5, $5, $3, $5, $1.5, $1.5, and 0, respectively. After 8:30am, the optimal time-varying prices increase with time, and with occupancy as well, except for Lot S-1 and S-2 with approximately constant price at $6. Between 8:30am and 9:30am, we would expect to price the Lots L-17/21/27/29 higher than in the UE case, so that a fraction of travelers can shift to use Lot L-18/20. After 9:30am, the SO pricing then under-prices Lots S-1 and S-2 to ensure they are used efficiently since they both are over-priced before 9:00am. At the end of the commuting time, all the lots are priced the same at $5.5. Note that the optimal parking prices resulted from the optimization problem 30 may not necessarily be smooth. The small spikes in the figure are due to the necessary pricing conditions between certain O-D pairs and certain time periods, to achieve the SO flow pattern. The smoothness can be further improved by adjusting the prices locally satisfying the required conditions 27, thanks to the fact that the SO prices are not unique.

Last but not least, the composite travel time cost, cruising time cost and total travel time cost of UE is $8,070, $1,751 and $9,821, respectively. The three numbers for SO are $7,766, $1,556 and $9,322, respectively. We see that the SO reduces both the composite travel time and cruising time considerably. The SO saves the total travel cost $498, by 5%. The savings are significant and the SO parking prices are efficient.
6. Conclusions

This paper investigates dynamic parking prices and provision of parking information to travelers for a general parking network. Travelers are aware of the real-time parking occupancy and real-time parking prices for all lots in the network. The real-time parking information helps traveler make time-varying parking choices to minimize their individual generalized travel cost. The dynamic parking prices and cruising (or searching) time are regarded as key components of the generalized travel cost. In the general parking network, each parking lot/block could be used by demands associated with multiple origins and multiple destinations.

We first formulate the parking choices under the UE conditions and solve it using the Variational Inequality (VI) approach. Given that the cruising time function is strictly monotone, we show that the UE flow pattern exists and is unique for any pre-determined time-varying parking prices. A projection-based algorithm is then proposed to solve the VI problem with guaranteed convergence to the UE solution. More importantly, we solve for the system optimal (SO) flow pattern and SO parking prices which minimizes the total system cost. The SO solutions involves three sequential steps, solving the optimal terminal parking occupancy, optimal total parking flow for the entire analysis time horizon, and the optimal parking flow pattern with respect to time. We show that the optimal terminal parking occupancies for the lots exist and
are unique, and they can be obtained by solving a standard traffic assignment problem in the general parking network. The SO parking flow pattern is, however, not unique. This offers sufficient flexibility for operators to achieve different management objectives while keeping the flow pattern optimal. The SO flow pattern is formulated as a large-scale linear program. Large-scale iterative algorithms methods for solving rectangular systems can be applied here to solve the linear program efficiently, such as LSQR or LSMR. Note that we do not explicitly model the roadway congestion when travelers choosing the parking lots, but the SO prices can be carefully chosen to produce constant arrival rates to each lot over time. This can reduce the potential roadway congestion and queuing due to cruising.

To achieve the optimal flow pattern, the optimal parking prices do not necessarily need to be path-based. Rather, any optimal flow pattern can be realized by a lot-based parking pricing scheme that only depends on the time or occupancy. Thus, the optimal parking prices are realistic and easy to implement in the real world. We further formulate the optimal parking prices as a large-scale linear program. Finally, the optimal parking prices with respect to the real-time occupancy can be therefore obtained.

Under the SO pricing, the change in the optimal parking price should go down as the occupancy increases (as long as the lot is used during that time period in its optimal flow pattern). This is because now providing occupancy information to travelers can also effect travelers’ parking choice, setting the price too high may otherwise prevent the facility being fully used. In addition, the optimal price change is equal to the marginal searching time of another lot being used during this time period. During this time interval, a lot with less occupancy is assigned with more travelers. We also show that under SO, any two parking lots cannot be used by travelers between more than one O-D pairs at any time. The only exception is that travelers’ perceived travel time difference between the most preferred two lots are approximately the same for any two O-D pairs.

We find that both the information provision and parking prices jointly serve as the efficient way of managing the traffic. The parking price alone may not be as much efficient, because the cruising time, as a key component of the generalized travel cost, is not affected by the optimal parking price alone. While the real-time occupancy (or the estimated cruising time) conveys the information of parking congestion level, it also effectively adjusts travelers parking choices in real time. With information provision, parking price is not the only economic manner to manage the traffic. The parking price and the information provision should be balance out and work jointly for the best result.

We finally apply our UE and SO formulations and solution algorithms to two numerical examples, a synthetic two-origin-two-destination network and a real parking network of Stanford University campus.
The results are intriguing. The best system performance is achieved by the parking prices such that the more preferred (convenient) lot should be used fully up to a certain terminal occupancy, meanwhile not too intensively. Around 5%-10% spaces in the preferred lot are expected to be available for the SO case, as a way to balance the cruising time and convenience of preferred lots. Those lots with less convenience should be used more efficiently under SO. In addition, the parking prices for the preferred lot are usually priced relatively high in the beginning of the commuting time, and later on priced relatively low as time goes on. This leads to mild (close to constant) arrival rates to the lots over the entire analysis horizon, which mitigates the potential queuing, cruising and roadway congestion.

Our future research will first focus on the calibration of UE model and searching time functions using real parking data on Stanford campus. We can furthermore consider more generic parking networks with explicit models of roadway congestions. Although this may not yield analytical formulations and solutions, it could be useful for designs of both roadway pricing and parking pricing. We are also interested in developing a variety of parking management objectives $F(\cdot)$ and pricing objectives $G(\cdot)$ for a more realistic implementation of the optimal pricing schemes. Time-varying step parking prices with respect to the occupancy would be of particular interest for parking operators.

References

Qian, Z. S., Rajagopal, R., 2012. Optimal dynamic pricing for morning commute parking with occupancy information. submitted to Transportation Research Part B.