

**Effect of Information Revelation Policies: A  
Game-Theoretic Analysis**

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## Abstract

Geographically dispersed sellers in electronic reverse-marketplaces such as those hosted by Freemarkets are uncertain about the number of competitors they face in any given market-session. Over the course of several market-sessions they learn about the competitive nature of the marketplace. How sellers learn to reduce the *market-structure uncertainty* depends on the market-transparency scheme, or the *revelation policy* adopted. A revelation policy determines the extent to which information such as the number of sellers in a session, their bidding patterns, etc are revealed to sellers. Since these policies control what sellers learn and how they bid in future sessions, these policies are a determinant of buyer surplus. Before the advent of the Internet, these revelation policies were not feasible, and studying the impact of these policies was irrelevant. Currently, there is little guidance available to buyers in choosing the appropriate information revelation policy.

To address this IT-enabled problem, we use game-theory to compare the buyer surplus generated under a set of revelation policies commonly used in electronic reverse marketplaces. We find that the policy that generates the least amount of market structure uncertainty for the sellers maximizes buyer surplus. This contradicts the traditional view that under uncertainty, sellers are worse off and buyers are better off. We investigate this apparent anomaly by comparing the behavior of sellers under market structure uncertainty to that under a framework where sellers are uncertain about their own costs. Based on our analysis, we find that it is the nature of the uncertainty, not the mere presence of uncertainty, that impacts buyer surplus.

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# 1 Introduction

FreeMarkets<sup>1</sup>, a successful B2B market-maker, convenes electronic reverse-auctions (*market-sessions*) at the request of buyers. Among other attributes, buyers determine the “market-transparency” scheme, or *information revelation policy*, to be used in the market-session. A revelation policy determines the nature of information about bids – winning bids, number of bidders etc. – that are revealed to geographically dispersed sellers in a market, at the beginning, in the middle and at the end of a market-session. At one end of the spectrum of available policies, the buyer can choose to accept sealed bids and inform each seller only about whether it won or lost in a given market-session. Under this policy, competitive information is not revealed to sellers. At the other end, the buyer can choose a revelation policy that allows sellers to observe the bids submitted by their opponents in real-time and react. Under this policy, sellers are aware of the number of other sellers, their bidding patterns and winning bid prices. Over the course of multiple market-sessions, the revelation policy adopted affects what sellers learn, how they bid in the future and the overall performance of the market including buyer surplus.

To our knowledge, this problem – the impact of these revelation policies when sellers are uncertain about the number of competitors (referred to as *market structure uncertainty*) – has not been studied in any prior work. Thus, little guidance is available for the buyer to choose the appropriate policy. One of the reasons that this problem has not been studied thus far could be that such revelation policies were not feasible to implement without the Internet or information technologies. For example, revelation policies such as the one where sellers are informed about their rank relative to other competitors are possible only because of the ease with such information can be computed and distributed. Given the current relevance and the need for such an analysis,

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<sup>1</sup><http://www.freemarkets.com>

we compare the impact of the following revelation policies using buyer surplus as our metric.

1. Complete-Market Structure Information Setting (CIS): All quotes are revealed to all participating sellers, implying that all sellers learn about the market structure.
2. Incomplete-Market Structure Information Setting (IIS): The only information revealed to participating sellers at the end of each market-session is the winner's quote. This means that the winning seller, which is already aware of its bid, does not learn anything about the market structure. However, losing bidders, if they exist, learn about the presence of at least one competitor and can bid accordingly in future market-sessions.

Note that these policies are but two of the many policies available to a buyer in a real-world marketplace such as those hosted by Freemarkets. Further, these specific policies are commonly adopted in both traditional marketplaces and electronic marketplaces (Thomas (1996)).

While our work is motivated by a real-world electronic marketplace hosted by Freemarkets, results discussed in this paper are applicable to any reverse-market setting that can create these different information regimes. Further, the results presented in this paper are actionable. We demonstrate that buyers benefit from choosing CIS over IIS so long as sellers value learning across market-sessions. In other words, the policy that generates the least amount of market structure uncertainty for the sellers maximizes buyer surplus. This contradicts the traditional view that under uncertainty, sellers are worse off and buyers are better off. We investigate this apparent anomaly by focusing on the nature of uncertainty. Specifically, we compare the behavior of sellers under market structure uncertainty to that under a framework where sellers are uncertain about their own costs. Based on our analysis, we demonstrate that it is the nature of the uncertainty, not the mere presence of uncertainty, that impacts buyer surplus.

The paper is organized in the following manner. Section 2 describes the problem context under the market structure uncertainty framework. In Section 3, we model the problem context described in Section 2 as a two period game. Results from these game-theoretic models are used to compare the buyer surplus generated in CIS and IIS. Next, we model the cost-structure uncertainty framework. Using the results from both frameworks, we demonstrate the differing nature of the impact of uncertainty on buyer surplus. Finally in Section 5, we conclude.

## **2 Problem Context**

We begin this section by describing a typical reverse-marketplace for coal. Buyers sequentially arrive at the electronic reverse-marketplace and initiate market-sessions. For each market-session convened at its request, the buyer is at the liberty of choosing its desired revelation policy. On the seller-side, there are a certain number of geographically distributed sellers who can offer coal. Only a subset of them bid in each market-session. For example, in each auction conducted by Freemarkets, a maximum of three or four coal sellers participate. Exogenous factors such as the type of coal the buyer wants or the distance between the coal mine and the buyer site limit seller participation. Participating sellers are unaware of their market structure (number of competitors) and learn about it across market-sessions. At the beginning of a market-session, each participating seller submits a multi-dimensional bid which includes coal content, ash content and water content of its coal, and the price quote. After reviewing all bids, the buyer chooses the best bid and awards the contract to the winner. At the end of the market-session, submitted bids are revealed according to the information revelation policy chosen by the buyer.

We portray this mechanism in our problem context in the following manner:

- There are exactly two buyers in the reverse-market. Each buyer initiates a market-session. This paper studies the impact of the revelation policy choice made by the first buyer. The second buyer is assumed to be unaware of the events of the previous market-session.
- There are only  $n$  sellers in the “world” providing the product and this is common knowledge. Exogenous factors influence seller participation and only a subset of them participate in the reverse-market (also referred to as *the market*). Thus the market can only be monopolistic<sup>2</sup>, duopolistic or competitive in nature. But participating sellers are unaware of the nature of the market. We model the market-structure uncertainty that each seller faces in the following manner. Let  $a$  s.t.  $0 < a < 1$  be the exogenous *participation probability* that each seller bids in the reverse-auction. All sellers are assumed to be aware of the value of  $a$  but they are unaware of the realized value of the participation probability for their opponents for that market-session<sup>3</sup>.

The market-mechanism for the reverse-auction is as described below:

- The auction conducted is a first price sealed bid “reverse”-auction.
- Each seller submits a price quote – a single dimensional bid which represents the multiple attributes (e.g. ash content, water content, etc.) of the bid. All participating sellers submit their bids without observing their opponent’s bid. Thus, it corresponds to a simultaneous move game. Each seller’s bid is based on the following:

– Its belief about market structure: In our set-up, sellers facing an uncertain number

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<sup>2</sup>During our discussion with Tony Bernhard, Technical Director, Freemarkets Inc., we learned about the possibilities of a monopolistic situation arising in their reverse-market context. This is the motivation for our assumption.

<sup>3</sup>For example when  $a = \frac{1}{2}$ , sellers know that nature tosses a coin for each seller and allows that seller to participate only if the outcome is a head. But each seller does not know if the outcome was a head or a tail when nature tossed a coin to decide whether or not to permit its opponent to participate in that market-session.

	Complete Information Setting (CIS)	Incomplete Information Setting (IIS)
Beginning of the first market-session	Uncertain	Uncertain
At the end of the first market-session	All bids are revealed	Only winner's bid revealed
Beginning of the second market-session	All sellers aware of market structure	<ul style="list-style-type: none"> <li>– Losing Seller is aware of the market structure.</li> <li>– Winner from the first market-session continues to be uncertain</li> </ul>

Table 1: Information Revealed under each Policy

of competitors hold symmetric beliefs about the market structure in the first market-session. Conditional on it being present in the reverse-auction, each seller holds a belief with probability  $a$  that each of its opponent is also present in the first market-session. We assume this to be common knowledge.

- Buyer utility for the product: The buyer utility for the product is  $U \in \mathbb{R}^+$  and it is assumed to be common knowledge.
- Cost of producing the product: All sellers are assumed to have identical costs,  $C \in \mathbb{R}^+$  and  $C < U$ . The value of  $C$  is also assumed to be common knowledge.
- Revelation Policy in the marketplace: It can be either IIS or CIS. Sellers are assumed to be aware of the revelation policy adopted.

Based on  $a, U, C$  and the policy adopted, sellers simultaneously bid prices.

After receiving all bids, the buyer chooses the seller that offers the lowest price as the winner (ties are broken randomly) and awards the contract. The winner builds the product but incurs a production cost,  $C$ . The product is delivered to the buyer, who in turn, remunerates the winner. This point corresponds to the end of one market-session. At this point, bids submitted in that market-

session are revealed according to the revelation policy adopted, allowing sellers to learn about the market-structure for the next market-session. Table 1 summarizes the information available to participating sellers at each instant under each policy. We model the seller participation for the second market-session in the following manner.

Anecdotal evidence from Freemarkets suggests that participation correlation across market-sessions is high i.e., more or less the same set of sellers repeatedly participate in each market-session.

- For the sake of tractability of the analytical model, we assume that participation is perfectly correlated from one market-session to the other. Stated differently, market structure from the first market-session is assumed to continue in future market-sessions.

With the same set of sellers as in the first market-session, the cycle is repeated for the second market-session. But note that beliefs which sellers hold about the market structure may be different in the second market-session than in the first market-session. Using game-theory, we study the impact of information revelation policy. Each market-session in our problem context maps to each period in our game-theoretic model.

### **3 Game-Theoretic Analysis**

In our model, each firm can choose any price i.e.,  $S \in \mathfrak{R}^+$ . However, note that no firm ever bids a price  $p > U$  or a price  $p < C$ . Thus, it suffices to consider bids in the range  $[C, U]$ . Without loss of generality, this range can be mapped and restricted so that the strategy set  $S \in [0, 1]$ .

This section is organized in the following manner. In subsection 3.1, we describe the single period game and characterize its equilibrium when (i) uncertainty is resolved and (ii) uncertainty

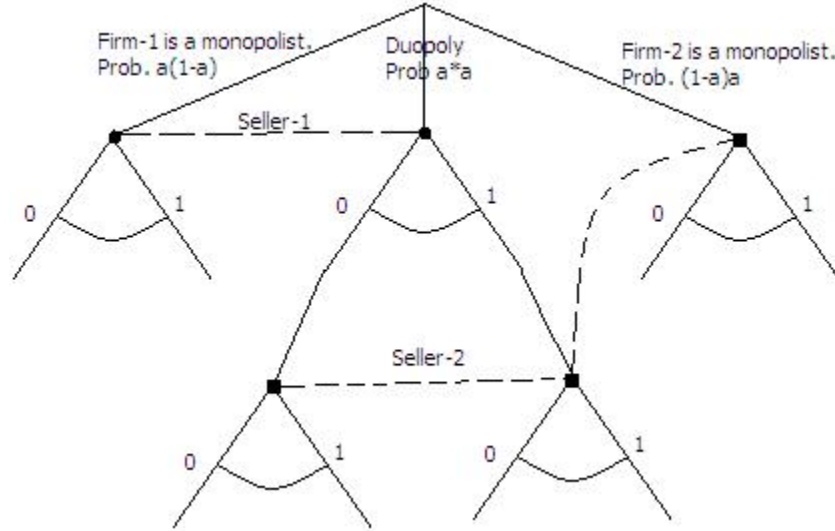


Figure 1: This tree depicts the extensive form of the single period game. Nature makes the first move (root-node), after which sellers move simultaneously (nonterminal nodes other than the root). The dotted lines denote each seller's information set, indicating that the seller that is to move at each node cannot distinguish between that node and the one to which it is connected.

persists. These equilibrium strategy calculations are extended to two period games under CIS and IIS frameworks in subsections 3.2 and 3.3 respectively.

### 3.1 Single Period Game

Consider a single period game where each seller is uncertain about the number of opponents it faces. The extensive form of this game is shown in Figure 1 for  $n = 1$  i.e., 2 seller case. The branches in the figure correspond to when: (left) both sellers are selected, probability  $a^2$ ; (center) seller-1 is selected but not seller-2, probability  $a(1 - a)$ ; (right) seller-2 is selected but not seller-1, probability  $(1 - a)a$ . When uncertainty is resolved, each seller knows the branch on which it lies: if a seller realizes that it is the only seller in the market, it extracts the buyer surplus by charging  $p = 1$ . Instead, if both sellers are aware of the presence of each other, both sellers bid  $p = 0$ .

With uncertainty, sellers are not aware of the branches on which they lie. But sellers hold

beliefs about being the only seller in the market. Based on this belief, they choose their bid prices to maximize profits. Since sellers are symmetrical, we restrict our attention to the symmetric Bayesian-Nash equilibrium.

A seller, say  $i$ , believes that it is the only seller in the market with a probability of  $(1 - a)^n$ . In this case, any bid  $p$  secures a profit of  $p$ . But with probability  $a(1 - a)^{n-1}$ , each of its  $\binom{n}{1}$  sellers can be present in the market. In this case,  $i$  secures a profit of  $p$  only if it outbids its opponent. Similarly, profits are calculated for all combinations of sellers.  $i$ 's expected profits  $\Pi_i$  as a function of seller  $i$ 's price  $p_i$  and all other sellers prices  $p_{\bar{i}}$  is:

$$\Pi_i(p_i, p_{\bar{i}}) = (1 - a)^n p_i + \binom{n}{1} a (1 - a)^{n-1} p_i \kappa_1 + \binom{n}{2} a^2 (1 - a)^{n-2} p_i \kappa_2 + \dots + a^n p_i \kappa_n$$

where

$$\kappa_k = \begin{cases} 1 & \text{if } p_i < \min(p_{\bar{i}}) \\ \frac{1}{k+1} & \text{if } p_i \text{ is bid by } k \text{ sellers} \\ 0 & \text{if } p_i > \min(p_{\bar{i}}) \end{cases}$$

**Proposition 3.1** *All sellers earn positive profits in equilibrium: i.e.,  $p_i^* = 0$  is not an equilibrium for any  $i$ .*

**Proof:** Suppose not: i.e., suppose  $p_i^* = 0$ . The proof proceeds by establishing the existence of  $p_i$  s.t.  $\Pi_i(p_i, p_{\bar{i}}^*) > \Pi_i(0, p_{\bar{i}}^*)$ . Choose  $p_i = 1$ . Now if  $p_{\bar{i}}^* = 1$ , then

$$\Pi_i(1, p_{\bar{i}}^*) = (1 - a)^n + \binom{n}{1} a (1 - a)^{n-1} \frac{1}{2} + \dots + a^n \frac{1}{n+1} > 0 = \Pi_i(0, p_{\bar{i}}^*)$$

Otherwise, if  $p_i^* < 1$ , then

$$\Pi_i(1, p_i^*) = (1 - a)^n > 0 = \Pi_i(0, p_i^*)$$

**QED**

The following proposition implies that there is no pure strategy equilibrium in the single period game. Intuitively, if all sellers bid a fixed price, one of the sellers can obtain from deviating. In turn, one among the other sellers can retaliate with yet a lower price. This retaliation process continues until a price  $\underline{p}$  is reached, at which point sellers stop retaliating and bid  $p = 1$ . The price  $\underline{p}$  is the point below which a seller is never incentivized to bid. It can obtain higher expected profits by bidding 1 rather than  $p < \underline{p}$ . But, if all sellers choose 1, then undercutting and retaliation will begin again. This implies that no pure strategy equilibrium exists.

**Proposition 3.2** *There is no pure strategy equilibrium.*

**Proof:** Suppose not: i.e., assume that there exists a pure strategy equilibrium  $(p_i^*, p_i^*)$ . Since sellers are identical, we focus only on the symmetrical equilibrium i.e.,  $p_i^* = p_i^*$ . The proof proceeds to establish the existence of  $p_i$  s.t.  $\Pi_i(p_i, p_i^*) > \Pi_i(p_i^*, p_i^*)$ .

$$\Pi_i(p_i^*, p_i^*) = (1 - a)^n p_i^* + \binom{n}{1} a (1 - a)^{n-1} \frac{p_i^*}{2} + \binom{n}{2} a^2 (1 - a)^{n-2} \frac{p_i^*}{3} + \dots + a^n \frac{p_i^*}{n} \quad (1)$$

Now if  $p_i = p_i^* - \epsilon$ , for some  $\epsilon > 0$ , then

$$\Pi_i(p_i, p_i^*) = (p_i^* - \epsilon) \quad (2)$$

From equations 1 and 2,  $\Pi_i(p_i, p_i^*) > \Pi_i(p_i^*, p_i^*)$  iff

$$\epsilon < \binom{n}{1} a(1-a)^{n-1} \frac{p_i^*}{2} + \binom{n}{2} a^2(1-a)^{n-2} \frac{2p_i^*}{3} + \dots + a^n \frac{(n-1)p_i^*}{n}$$

Such an  $\epsilon$  exists, since  $(1-a) > 0$  and  $p_i^* > 0$ , by Proposition 3.1. **QED**

Having argued that no pure strategy equilibrium exists, we now derive the mixed strategy Bayesian-Nash equilibrium of the single period game. The equilibrium strategy for seller  $i$  is given by a pair  $(f_i, S_i)$ , where  $f_i$  is a probability measure defined over the strategy set  $S_i \subseteq \mathcal{S}$  (i.e.,  $S_i$  is the support of  $f_i$ ), such that  $(f_i, S_i)$  is a best response to equilibrium strategy of any of its opponent  $j \in \bar{i}$  and  $i \notin \bar{i}$ . The nature of the solution is characterized by the propositions below.

Assuming that each of  $i$ 's opponents bids according to cumulative probability distribution  $F_{\bar{i}}(p)$ , then the probability that  $i$  outbids (i.e., undercuts) each of its opponents is given by  $(1 - F_{\bar{i}}(p))$ . Thus, seller  $i$ 's expected profits  $\Pi_i$  in terms of its bid price  $p$  are calculated as follows:

$$\begin{aligned} \Pi_i(p) = & \overbrace{(1-a)^n p}^{\text{No opponent}} + \overbrace{\binom{n}{1} (1-a)^{n-1} a (1-F_{\bar{i}}(p)) p + \dots +}^{\text{1 opponent}} \\ & \underbrace{\binom{n}{k} (1-a)^{n-k} a^k (1-F_{\bar{i}}(p))^k p}_{k \text{ opponents}} + \dots + \underbrace{a^n (1-F_{\bar{i}}(p))^n p}_{n \text{ opponents}} \end{aligned}$$

Simplifying this expression, we get

$$\Pi_i(p) = ((1-a) + a(1-F_{\bar{i}}(p)))^n p \quad (3)$$

**Proposition 3.3** *There are no gaps in the strategy sets  $S_i$  and  $S_{\bar{i}}$ .*

**Proof:** Since sellers are symmetrical, so are the strategy sets. Let a hole exist in the interval

$I = (h_{\min}, h_{\max})$  in the strategy set of all sellers.

The profit for any seller  $i$  when it charges  $h_{\min}$  is

$$\Pi_i(h_{\min}) = ((1 - a) + a(1 - F_{\bar{i}}(h_{\min})))^n h_{\min}$$

The profit for seller  $i$  when it charges  $h_{\max}$  is

$$\Pi_i(h_{\max}) = ((1 - a) + a(1 - F_{\bar{i}}(h_{\max})))^n h_{\max}$$

Since  $F_{\bar{i}}(h_{\max}) = F_{\bar{i}}(h_{\min})$ ,  $\Pi_i(h_{\max}) > \Pi_i(h_{\min})$  which violates the definition of mixed strategy equilibrium. **QED**

**Proposition 3.4** *No seller has mass point a) in the interior or b) at the lower boundary or c) at the upper boundary of the other seller's support if that boundary has a mass point for the other seller.*

**Proof:** Let us represent  $\underline{p}_i = \inf(S_i)$  and  $\bar{p}_i = \sup(S_i)$ . Assume on the contrary that there exists a mass point in equilibrium for any seller  $j \in \bar{i}$  at the price  $\underline{p}_i < p_i^* < \bar{p}_i$  with probability of  $\gamma > 0$ .

By Proposition 3.3, there are no “holes” in the strategy set  $S_i$ . Thus, we can consider the profits for seller  $i$  when bidding  $p_i^* - \epsilon$  and  $p_i^* + \epsilon$ , for  $\epsilon > 0$ , namely:

$$\Pi_i(p_i^* - \epsilon) = ((1 - a) + a(1 - F_{\bar{i}}(p_i^* - \epsilon)))^n (p_i^* - \epsilon) \tag{4}$$

$$\Pi_i(p_i^* + \epsilon) = ((1 - a) + a(1 - F_{\bar{i}}(p_i^* + \epsilon)))^n (p_i^* + \epsilon) \tag{5}$$

Subtracting equation 4 from equation 5 yields

$$p_i^* \left( (1 - a) + a(1 - F_{\bar{i}}(p_i^* - \epsilon)) \right)^n - \left( (1 - a) + a(1 - F_{\bar{i}}(p_i^* + \epsilon)) \right)^n -$$

$$\epsilon((1-a) + a(1 - F_{\bar{i}}(p_i^* - \epsilon)))^n + ((1-a) + a(1 - F_{\bar{i}}(p_i^* + \epsilon)))^n$$

This difference is strictly greater than zero only for  $\epsilon$  satisfying the condition

$$0 < \epsilon < \frac{p_{\bar{i}} \left[ (1-a) + a(1 - F_{\bar{i}}(p_i^* - \epsilon))^n - ((1-a) + a(1 - F_{\bar{i}}(p_i^* + \epsilon)))^n \right]}{\left[ (1-a) + a(1 - F_{\bar{i}}(p_i^* - \epsilon))^n + ((1-a) + a(1 - F_{\bar{i}}(p_i^* + \epsilon)))^n \right]} \quad (6)$$

This suggests that  $i$  earns strictly greater profits by shifting some mass from above  $p_i^*$  to below  $p_i^*$ .

Thus,  $j \in \bar{i}$  cannot have a mass point in the interior of seller  $i$ 's support at equilibrium.

If instead,  $i$  has a mass point at  $\bar{p}_i$  with probability  $\gamma_1$  and  $j$  continues to have a mass point at  $p_i^* = \bar{p}_i$  with probability  $\gamma$ , then seller  $i$  can do better by bidding  $\bar{p}_i - \epsilon$  with probability  $\gamma_1$  and  $p_i^*$  with zero density. The only case when the proof does not hold good, is when  $p_i^* = \bar{p}_i$  and seller  $i$  does not have mass point at  $\bar{p}_i$ . **QED**

**Corollary 3.5** *If one seller  $i$  has a mass point at  $\bar{p}_i = \sup(S_i)$ , all other sellers will charge  $\bar{p}_i$  with zero density i.e., any other seller  $j \in \bar{i}$  will randomize in the interval  $[p, \bar{p}_i)$ .*

**Proof:** Suppose seller  $i$  charges  $p_i^* = \bar{p}_i$  with probability  $\gamma$ , consider the profits for any other seller  $j \in \bar{i}$  from charging  $p_i^* - \epsilon$  and  $p_i^*$ . These are respectively

$$\Pi_j(p_i^* - \epsilon) = [(1-a) + a(1 - F_i(p_i^* - \epsilon))] ((1-a) + (1 - F_i(p_i^* - \epsilon))a)^{n-1} (p_i^* - \epsilon)$$

$$\Pi_j(p_i^*) = (1-a)((1-a) + (1 - F_i(p_i^*))a)^{n-1} (p_i^*) + a((1-a) + (1 - F_i(p_i^*))a)^{n-1} \gamma \frac{p_i^*}{2}$$

Subtracting  $\Pi_j(p_i^*)$  from  $\Pi_j(p_i^* - \epsilon)$ , the difference is

$$\approx \left( a \frac{\gamma p}{2} - a\epsilon\gamma - x\epsilon \right) ((1-a) + (1 - F_i(p_i^*))a)^{n-1} \quad (7)$$

This is strictly greater than zero for sufficiently small  $\epsilon$  implying that seller  $j$  can secure higher profits by bidding a value  $\epsilon$  less than  $p_i^*$  and therefore the proof. **QED**

**Proposition 3.6**  $\sup(S_i) = 1$ .

**Proof:** Suppose not: i.e., suppose  $\sup(S_i) = \bar{p}_i < 1$ . By Proposition 3.4,  $F_{\bar{i}}(\bar{p}_i) = 1$ . Thus,  $\Pi_i(\bar{p}_i) = (1 - a)^n \bar{p}_i < (1 - a)^n = \Pi_i(1)$ . Therefore,  $\bar{p}_i = 1$ . **QED**

Based on these propositions, one can compute the mixed strategy equilibrium for any seller  $j \in \bar{i}, i \notin \bar{i}$  as that distribution  $f_i$  over the strategy set  $S_i$  which generates the same profits for any seller  $i$  independent of  $i$ 's strategy. More specifically,  $i$ 's expected profits at any price  $p$  must equal that at the upper boundary of  $S_i$ , namely  $p = 1$ . Since  $F_{\bar{i}}(1) = 1$ ,  $i$ 's expected profits equal  $(1 - a)^n$  at this boundary from equation 3. But this profit is the same for any price  $p$ . Therefore,

$$(1 - a)^n = ((1 - a) + a(1 - F_{\bar{i}}(p)))^n p$$

Rearranging,

$$F_{\bar{i}}(p) = 1 - \left[ \left( \frac{1}{p} \right)^{\frac{1}{n}} - 1 \right] \frac{(1 - a)}{a} \quad (8)$$

Corresponding to this cdf, the pdf is

$$f_{\bar{i}}(p) = \frac{(1 - a)}{n p^{\frac{1}{n} + 1} a} \quad (9)$$

Since we assume a symmetric equilibrium,  $f_{\bar{i}}(p) \equiv f_i(p)$  and  $F_{\bar{i}}(p) \equiv F_i(p)$ .

**Proposition 3.7**  $\inf(S_i) = (1 - a)^n$ .

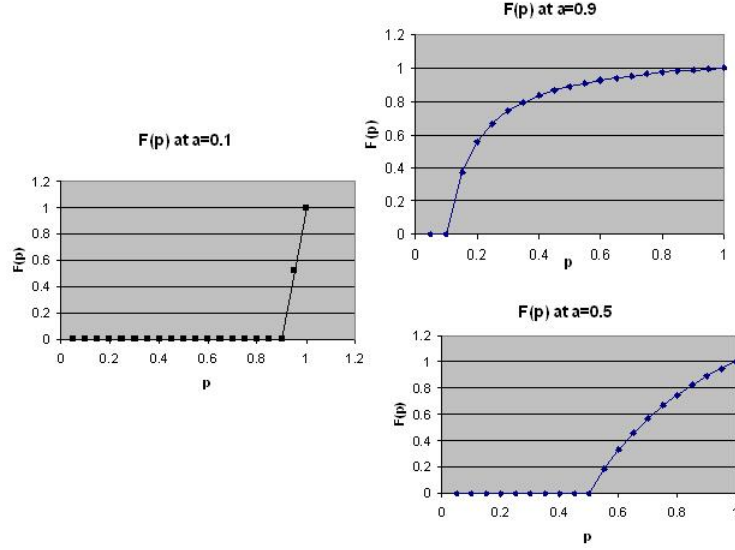


Figure 2: CDF for different values of  $a$  for  $n = 1$  i.e., the 2-seller-case.

**Proof:** Let  $F_i(p) = 0$ . **QED**

Summarizing the equilibrium bid distribution for the single period game: sellers play according to the cdf given in equation 8 at equilibrium. This cdf is plotted in figure 2 for  $n = 1$  and  $a \in \{0.9, 0.5, 0.1\}$ . Also since  $F_i(1) = 1$ , there is no mass point at  $p = 1$ .

### 3.2 Complete Information Setting

The single period game discussed earlier is extended to a two period game under the CIS framework (See Figure 3). By observing the number of opponents that bid  $p \leq 1$ , sellers can learn about the the number of sellers for the second period game.

With uncertainty resolved, if a seller realizes that it is the only seller in the market, it bids  $p = 1$  in the second period. However, if there are more than one seller in the market and each is aware of the existence of the other, they bid  $p = 0$  at equilibrium. Based on this, we can calculate the expected (minimum) price  $\hat{p}_{CIS}^2$  observed by the second buyer conditional on having at least one

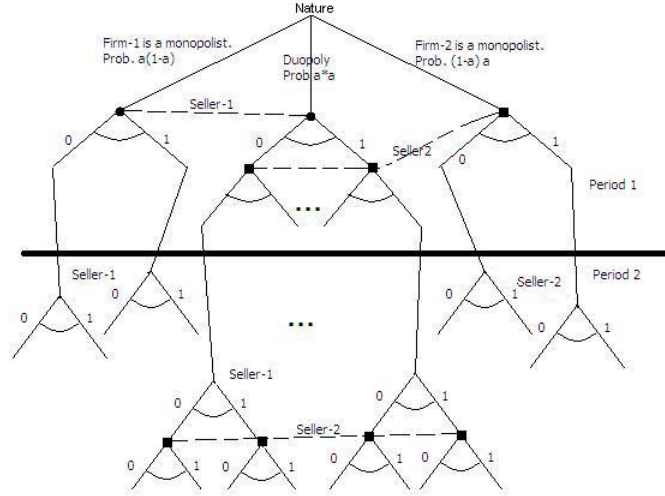


Figure 3: Complete Information Setting. This tree depicts the extensive form of the two period game in the CIS. sellers are uncertain about the market structure in the first period. In the second period, sellers are certain about the market structure.

seller in the market. With probability of  $a(1 - a)^n$ , each of the  $(n + 1)$  sellers realizes that it is the only seller in the market. Therefore,

$$\hat{p}_{\text{CIS}}^2 = (n + 1)a(1 - a)^n \quad (10)$$

The second period expected profits for seller  $i$  are

$$\Pi_{\text{CIS}}^{i2} = a(1 - a)^n \quad (11)$$

This expression can be explained in the following manner: If  $i$  is the only seller in the market, its profits are 1; alternatively, if the market has more than one seller, each seller earns 0 profits.

Note that these second period expected profits are independent of whether or not the seller won the first period. In other words, the second period expected profits are dependent only on the second period game and not on the outcome of the first period game. It follows that the equilibrium

strategies in the first period are also dependent only on the profits earned during that period.

**Lemma 3.8** *Under CIS, the first period Bayesian-Nash equilibrium is equivalent to that of the single period game.*

**Proof:** If  $F_{\text{CIS}}^{\bar{i}1}$  is the first period bid distribution for seller  $j \in \bar{i}$ , then  $i$ 's total expected profits from both periods are

$$\begin{aligned}
\Pi_{\text{CIS}}^{i1}(p) &= (1-a)^n(p + \Pi_{\text{CIS}}^{i2}) \\
&+ \binom{n}{1} (1-a)^{n-1} a \left( (1 - F_{\text{CIS}}^{\bar{i}1}(p))(p + \Pi_{\text{CIS}}^{i2}) + F_{\text{CIS}}^{\bar{i}1}(p)(0 + \Pi_{\text{CIS}}^{i2}) \right) + \dots \\
&\quad \underbrace{\hspace{15em}}_{k \text{ other opponents}} \\
&+ \binom{n}{k} (1-a)^{n-k} a^k \left( \overbrace{(1 - F_{\text{CIS}}^{\bar{i}1}(p))^k}^{\text{seller } i \text{ wins}} \left( \underbrace{p}_{\text{1st period}} + \underbrace{\Pi_{\text{CIS}}^{i2}}_{\text{2nd period}} \right) + \overbrace{(1 - (1 - F_{\text{CIS}}^{\bar{i}1}(p))^k)}^{\text{one among the other } k \text{ sellers wins}} \left( \underbrace{0}_{\text{1st period}} + \underbrace{\Pi_{\text{CIS}}^{i2}}_{\text{2nd period}} \right) \right) \\
&+ \dots + a^n \left( (1 - F_{\text{CIS}}^{\bar{i}1}(p))^n (p + \Pi_{\text{CIS}}^{i2}) + (1 - (1 - F_{\text{CIS}}^{\bar{i}1}(p))^n) (0 + \Pi_{\text{CIS}}^{i2}) \right)
\end{aligned}$$

Rearranging the terms, we have

$$\Pi_{\text{CIS}}^{i1}(p) = \Pi_{\text{CIS}}^{i2} + ((1-a) + a(1 - F_{\text{CIS}}^{\bar{i}1}(p)))^n p \tag{12}$$

Note that this simply means that the expected profit of the combined game for any price, is the expected profit of the single period game for the same price, incremented by a constant  $\Pi_{\text{CIS}}^{i2}$ . Recall that if a constant is added or subtracted from the payoffs for all prices, the equilibrium of this new game is equivalent to that of the unaltered game. Based on this well-known lemma, we prove that the equilibrium of the first period game is equivalent to that of the single period game. **QED**

Using Lemma 3.8, we can write the first period equilibrium strategy for  $j \in \bar{i}$  as the following

cdf:

$$F_{\text{CIS}}^{\bar{i}1} = 1 - \left[ \left( \frac{1}{p} \right)^{\frac{1}{n}} - 1 \right] \frac{(1-a)}{a} \quad (13)$$

Its corresponding pdf is:

$$f_{\text{CIS}}^{\bar{i}1} = \frac{(1-a)}{n p^{\frac{1}{n}+1} a} \quad (14)$$

Since the game is symmetric,  $f_{\text{CIS}}^1 \equiv f_{\text{CIS}}^{\bar{i}1} \equiv f_{\text{CIS}}^{i1}$  for all sellers in the market.

The expected minimum price observed by the buyer in the first period of the CIS, conditional on having at least seller in the reverse-auction, is:

$$\hat{p}_{\text{CIS}}^1 = \sum_{k=1}^{n+1} \left[ (1-a)^k a^{n-k+1} \binom{n+1}{k} \right] \left\{ \int_{(1-a)^n}^1 [k (1 - F_{\text{CIS}}^1(p))^{k-1} f_{\text{CIS}}^1(p)] p dp \right\} \quad (15)$$

In this expression, the term in the second square bracket represents the probability distribution of the minimum price observed when there are  $k$  sellers in the market. The first square brackets represent the probability of having  $k$  sellers in the market.

Having computed  $\hat{p}_{\text{CIS}}^1$  and  $\hat{p}_{\text{CIS}}^2$ , the total expected minimum price in the CIS is given by:

$$\hat{p}_{\text{CIS}} = \hat{p}_{\text{CIS}}^1 + \hat{p}_{\text{CIS}}^2 \quad (16)$$

### 3.3 Incomplete Information Setting

Recall that in the first period of IIS, sellers are unaware of the number of competitors they face. But in the second period, losing sellers from the first period, if at least one exists, learn that about the presence of at least one competitor. However, the winner remains unaware of the nature of competition. The extensive form game for  $n = 1$  is shown in figure 4.

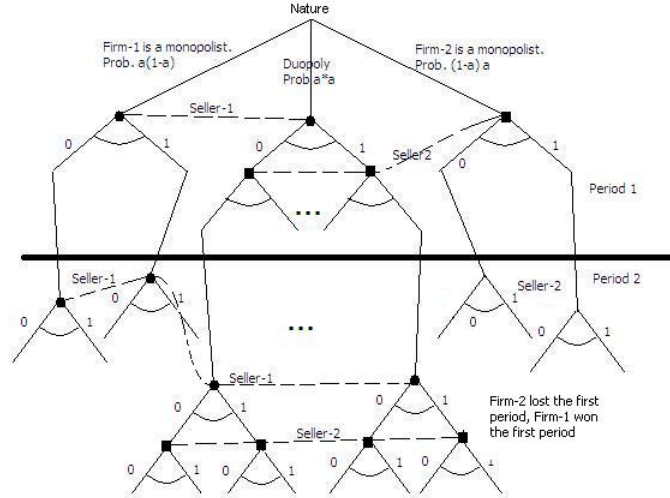


Figure 4: Incomplete Information Setting for  $n = 1$ . This tree depicts the extensive form of the two period game in the IIS. Sellers are uncertain about the number of competitors in the first period. In the second period, one seller is the winner and the other seller is aware of the existence of at least one competitor.

In this section, we characterize the Bayesian-Nash mixed strategy for the sellers under IIS using propositions. Most proofs for these propositions are similar to the ones in the single period game. The key difference between these two sets of proofs is that in IIS, sellers hold asymmetric beliefs about being the only seller in the market. Specifically, the loser(s) from the first period, have their second period belief about being a the only seller as 0. But the winner from the first period, has a belief of  $x'^n > 0$ . This  $x'$  represents the second period belief held by winner that each of the other sellers are not present in the market. Given this, we begin with the second period game and determine the equilibrium based on the following propositions

### 3.3.1 Second Period Game

**Proposition 3.9** *There is no equilibrium in pure strategies in the second period.*

Since there is no pure strategy equilibrium, let us represent the second period equilibrium as a pair  $(f_l, S_l)$  for the loser(s) and  $(f_w, S_w)$  for the winner. The second period expected profits for the

first period winner are given by:

$$\Pi_w^2(p) = x^n p + \binom{n}{1} (1-x') x'^{n-1} (1-F_l(p)) p + \dots + a^n (1-F_l(p))^n p \quad (17)$$

where  $F_l(p)$  represents the cdf of loser's bid distribution. But from the perspective of the loser, at least one seller exists. Conditional on this information, the loser's expected profits are:

$$\begin{aligned} \Pi_l^2(p) = & \frac{\binom{n}{1} (1-a)^{n-1} a}{(1-(1-a)^n)} (1-F_w(p)) p + \dots + \frac{\binom{n}{k} (1-a)^{n-k} a^k}{(1-(1-a)^n)} (1-F_w(p)) (1-F_l(p))^{k-1} p \\ & + \dots + \frac{a^n}{(1-(1-a)^n)} (1-F_w(p)) (1-F_l(p))^{n-1} p \end{aligned} \quad (18)$$

where  $F_w(p)$  represents the cdf of the winner's bid distribution. Note that the losing seller knows, with certainty, that the market has at least one competitor. In this framework, the equilibrium solution can be further characterized by the following propositions:

**Proposition 3.10** *Strategy sets are identical for both the winner and the loser sellers.*

**Proposition 3.11** *Neither type – winner or loser – has mass points a) in the interior or b) at the lower boundary of the other's support or c) at the upper boundary of other's support if that boundary has a mass point for the other seller.*

**Proposition 3.12** *If one type – winner or loser – has a mass point at  $\bar{p}$ , the other type charges  $\bar{p}$  with zero density in equilibrium.*

**Proposition 3.13**  $\sup(S_w^*) = \sup(S_l^*) = 1$ .

By definition, the loser's mixed equilibrium strategy has to generate the same expected profits for its opponent independent of the opponent's action i.e., the loser's equilibrium strategy has to

generate the same expected profits for the winner as it would when the winner bids  $p = 1$ . But

$\Pi_w^2(1) = x'^n$ , since  $F_l(1) = 1$ . Thus,

$$\Pi_w^2(p) = (x' + (1 - x')(1 - F_l(p)))^n p = x'^n \quad (19)$$

Rearranging terms,

$$F_l(p) = 1 - \left[ \left( \frac{1}{p} \right)^{\frac{1}{n}} - 1 \right] \frac{x'}{(1 - x')} \quad (20)$$

Corresponding to this cdf is the pdf

$$f_l(p) = \frac{x'}{n p^{\frac{1}{n}+1} (1 - x')} \quad (21)$$

Observe that equation 20 is similar to the equilibrium distribution for the single period game given by equation 8. From this, one can also characterize the lower boundary of the loser's strategy set as well as the winner's strategy set.

**Proposition 3.14**  $\inf(S_w^*) = \inf(S_l^*) = x'^n$ .

**Proof:** To determine  $\inf(S_l^*)$ , set  $F_l(p) = 0$ . Based on that, we find  $\inf(S_l^*) = x'^n$ . We still need to prove that  $\inf(S_w^*) = \inf(S_l^*)$ . Suppose not: i.e.,  $\inf(S_w^*) \neq \inf(S_l^*)$

Case 1: Let  $\underline{p} = \inf(S_w^*) < \inf(S_l^*) = x'^n$ . In such a case, since losers always bid a price  $p \geq x'^n$ , the expected profits for the winner can be increased by increasing  $\underline{p}$  to  $\inf(S_l^*)$ . This proves that  $\inf(S_w^*) \geq \inf(S_l^*)$ .

Case2: Let us consider the second case -  $\underline{p} = \inf(S_w^*) > \inf(S_l^*)$ . In such a case, the loser can seek higher profits by shifting its  $\inf(S_l^*)$  to  $\underline{p}$ . This would mean that it has a mass point at  $\underline{p}$ . But from proposition 3.11, we know that there can be no mass point at the lower boundary of the

other seller's support. Therefore, the only possibility is  $\inf(S_w^*) = \inf(S_l^*)$ . This combined with our earlier portion of this proof  $\inf(S_l^*) = x'^n$  proves the proposition. **QED**

Using proposition 3.14 and the definition of the mixed strategy equilibrium, we compute the cdf of the winner's bid distribution. By definition, the mixed strategy equilibrium for the winner has to generate the same expected profits for the loser independent of the price the loser bids i.e., the winner's strategy has to generate the same profit for the loser at any  $p$  as that when the loser bids the  $p = \inf(S_l^*)$ . Thus, one can equate equation 18 to the profits earned by bidding  $p = \inf(S_l^*) = x'$ :

$$\frac{(\sum_{k=1}^n \binom{n}{k} (1-a)^{n-k} a^k (1-F_l(p))^{k-1})}{(1-(1-a)^n)} (1-F_w(p))p = x'^n \quad (22)$$

It follows that

$$F_w(p) = 1 - \frac{x'^n (1-(1-a)^n)}{\left[ \sum_{k=1}^n \binom{n}{k} (1-a)^{n-k} a^k (1-F_l(p))^{k-1} \right] p} \quad (23)$$

At  $p = 1$ , the winner has a mass point of  $M_w = \frac{x'^n(1-x'^n)}{n(1-a)^{n-1}(1-x')}$ . This mass point at  $p = 1$  ensures that the distribution  $F_w(p) = 1$  at  $p = 1$ .

Corresponding to this cdf, the pdf is

$$f_w(p) = \frac{x'^n (1-(1-a)^n)}{\left[ \sum_{k=1}^n \binom{n}{k} (1-a)^{n-k} a^k (1-F_l(p))^{k-1} \right] p^2} \quad (24)$$

Note that since the winner continues to be uncertain about the number of sellers in the market, the winner's equilibrium distribution  $F_w(p)$  in the second period is the same, regardless of whether the market has one sellers or more. Also note that the winner's bid distribution is first-degree stochastic

dominant over losers’.

Using  $F_w(p)$  and  $F_l(p)$ , we compute the expected paid by the second buyer conditional on having at least one seller in the reverse-auction as:

$$\begin{aligned} \hat{p}_{\text{us}}^2 &= \sum_{k=1}^{n+1} \left[ \binom{n+1}{k} a^k (1-a)^{n-k+1} \int_{x^n}^1 \right] \\ &\quad \left[ (1 - F_l(p))^{k-1} f_w(p) + (k-1)(1 - F_l(p))^{k-2} (1 - F_w(p)) f_l(p) \right] p dp \\ &\quad + (n+1)a(1-a)^n M_w \end{aligned}$$

In this expression, the term inside the square brackets in the first line represents the probability that  $k$  sellers are in the market. In the second line, the term inside the square brackets represents the probability distribution of the lowest price bid when  $k$  sellers including one winner are in the market. The first term in last line accounts for the mass point that is observed when the winner is the only seller in the market.

**Lemma 3.15** *The second period expected profits for the winner and the loser are the same and are equal to  $x^n$ .*

### 3.3.2 First Period Game

Recall that  $x^n$  is the second period belief held by the first-period winner that none of its opponents is present. Thus, the probability of that the first-period winner is the only seller in the market,

$$x^n = \frac{\text{Probability of being the only seller}}{\text{Probability of winning the first period}}$$

Note that the probability of winning the first period is dependent on the equilibrium of the first period game. Using this definition, we characterize the equilibrium for the first period game.

If the first period had a pure strategy equilibrium (since sellers are identical, we focus only the symmetrical equilibrium):

$$x' = \left( \frac{(1-a)^n}{\sum_{k=0}^n \binom{n}{k} (1-a)^{n-k} a^k \frac{1}{k+1}} \right)^{\frac{1}{n}} \quad (25)$$

Given this:

**Proposition 3.16** *There is no equilibrium in pure strategies in the first period.*

**Proof:** The proof first assumes the existence of a pure equilibrium and identifies the point beyond which price war will not continue. In the second step, the proof will demonstrate that the lowest price below which price war does not continue is not in the optimal strategy set for a seller. Based on these two steps, we conclude that only a mixed strategy equilibrium exists.

Let the symmetrical pure strategy equilibrium price be  $p^* = p_i^* = p_i^*$ . The combined expected profits for any seller  $i$  in such a case is

$$\begin{aligned} \Pi_i(p_i^*, p_i^*) &= (1-a)^n [p^* + x'^n] + \binom{n}{1} (1-a)^{n-1} a \left[ \frac{1}{2} (p^* + x'^n) + \frac{1}{2} (0 + x'^n) \right] \\ &+ \dots + \overbrace{\binom{n}{k} (1-a)^{n-k} a^k}^{\text{Probability that } k \text{ opponents are present}} \left[ \overbrace{\frac{1}{k+1} (p^* + x'^n)}^{\text{Seller } i \text{ wins}} + \overbrace{\frac{k}{k+1} (0 + x'^n)}^{\text{One of the } k \text{ competitors wins}} \right] \\ &+ \dots + a^n \left[ \frac{1}{n+1} (p^* + x'^n) + \frac{n}{n+1} (0 + x'^n) \right] \end{aligned}$$

$i$ 's profits when it deviates from the equilibrium by undercutting the price bid by others is

$$\Pi_i(p_i^* - \gamma, p_i^*) = p^* - \gamma + (1-a)^n \quad (26)$$

Based on these two expressions, we can state that  $i$  will undercut so long as it can find a  $\gamma > 0$  and if

$$\Pi_i(p_i^* - \gamma, p_i^*) > \Pi_i(p_i^*, p_i^*) \quad (27)$$

Substituting the expression for  $x^m$  and rearranging the terms, we can say that price war continues as long as  $\gamma$  satisfies:

$$0 < \gamma < \left( p^* - \frac{(1-a)^n}{\sum_{k=0}^n \binom{n}{k} (1-a)^{n-k} a^k \frac{1}{k+1}} \right) \left( 1 - \sum_{k=0}^n \binom{n}{k} (1-a)^{n-k} a^k \frac{1}{k+1} \right)$$

Price war will cease to exist at the point when  $\gamma = 0$  i.e.,

$$p_{pure}^* = \frac{(1-a)^n}{\sum_{k=0}^n \binom{n}{k} (1-a)^{n-k} a^k \frac{1}{k+1}}$$

Note that this  $p_{pure}^* < 1$  is the pure strategy equilibrium if one exists. We can be rest assured that a pure equilibrium does not exist, if we prove that  $\Pi_i(1, p^*) > \Pi_i(p_{pure}^*, p_{pure}^*)$ . Such a proof implies that a price  $p > p_{pure}^*$  exists below which the seller  $i$  will not undercut but perceives higher expected profits by bidding  $p = 1$ . For this proof, we compute seller  $i$ 's expected pure strategy equilibrium profit as

$$\Pi_i(p_{pure}^*, p_{pure}^*) = (1-a)^n + \frac{(1-a)^n}{\sum_{k=0}^n \binom{n}{k} (1-a)^{n-k} a^k \frac{1}{k+1}} \quad (28)$$

Similarly, we compute seller  $i$ 's expected profit from deviating to  $p = 1$

$$\Pi_i(1, p^*) = 2(1-a)^n + \sum_{m=1}^n \binom{n}{m} (1-a)^{n-m} a^m \frac{(1-a)^n}{\sum_{k=0}^n \binom{n}{k} (1-a)^{n-k} a^k \frac{1}{k+1}} \quad (29)$$

Based on these expressions, we can say that if  $\Pi_i(1, p^*) > \Pi_i(p_{pure}^*, p_{pure}^*)$  no pure strategy equilibrium exists. Substituting for the expressions from equations 28 and equation 29, we can say that the comparison becomes

$$1 - p_{pure}^* > 0 \quad (30)$$

This is always true and therefore, the proof. **QED**

Given that no pure strategy equilibrium exists in the first period, let us define the mixed strategy equilibrium for each seller as the distribution  $(f_{\text{IS}}^1, S_{\text{IS}}^1)$  and its corresponding cdf as  $F_{\text{IS}}^1(p)$ . Using this, we can define for  $x'$ .

If a seller, say  $i$ , wins in the first period, it is because either (a) it is a monopolist, or (b) it has one opponent and it outbid the opponent with probability  $(1 - F_{\text{IS}}^{\bar{i}1}(p))$ , where  $F_{\text{IS}}^{\bar{i}1}(p) \equiv F_{\text{IS}}^{i1}(p) \equiv F_{\text{IS}}^1(p)$  is its opponent's bid distribution. The one opponent can be any of the  $\binom{n}{1}$  sellers. (c) it has two opponents and it outbid them with probability  $(1 - F_{\text{IS}}^{\bar{i}1}(p))^2$  and the two opponents can be any of the  $\binom{n}{2}$  combinations and so on. Thus, for seller  $i$ , its second period belief  $x'_i$ , conditioned on winning in the first period with a price  $p$ , can be represented in a Bayesian manner as follows:

$$x'_i(p) = \left( \frac{(1-a)^n}{(1-a)^n + \binom{n}{1} (1-a)^{n-1} a (1 - F_{\text{IS}}^{\bar{i}1}(p)) + \dots + a^n (1 - F_{\text{IS}}^{\bar{i}1}(p))^n} \right)^{\frac{1}{n}} \quad (31)$$

Recall that  $(1-a)^n$  in this expression is seller  $i$ 's first period belief about being a monopolist.

Simplifying this,

$$x'_i(p) = \left[ \frac{(1-a)}{(1-a) + a(1 - F_{\text{IS}}^{\bar{i}1}(p))} \right] \quad (32)$$

Recall that  $x'^n$  is also the second period profits. Using this definition of  $x'$ ,  $i$ 's expected profit

across both periods are

$$\begin{aligned}
\Pi_i(p_i) = & \overbrace{\left[ (1-a) + a(1 - F_{\text{IS}}^{\bar{i}1}(p_i)) \right]^n \left( \underbrace{p_i}_{\text{1st Period Profit}} + \underbrace{x'_i(p_i)}_{\text{2nd Period Profit}} \right)}^{\text{Seller } i \text{ wins the first period}} + \\
& \overbrace{\sum_{k=1}^n \binom{n}{k} (1-a)^{(n-k)} a^k \left[ \{1 - (1 - F_{\text{IS}}^{\bar{i}1}(p_i))^k\} x_{i|k}^{\prime n}(p_{\bar{i}}) \right]}^{\text{Seller } i \text{ loses the first period. This corresponds to the second period profits only.}} \quad (33)
\end{aligned}$$

where  $x_{i|k}^{\prime n}(p_{\bar{i}})$  is the second period belief held by the winner (not seller  $i$ ) about being a monopolist conditioned on  $k < n$  sellers in the market. In this expression, the first line represents the expected profits when seller  $i$  wins. This happens with a probability of  $\left[ (1-a) + a(1 - F_{\text{IS}}^{\bar{i}1}(p_i)) \right]^n$ . In which case, it secures a profit of  $p_i$  in the first period. It updates its belief about being the only seller to  $x'(p_i)^n$  for the second period and secures an expected profit of  $x'(p_i)^n$ . The second line corresponds to the condition when seller  $i$  loses the first period. If it loses when  $k$  other sellers are present in the market, the probability that it loses is  $\{1 - (1 - F_{\text{IS}}^{\bar{i}1}(p_i))^k\}$ . Under this case, the expected profit is equal to the belief held by the winner which is represented by  $x_{i|k}^{\prime n}(p_{\bar{i}})$ .

Note that although  $x_{i|k}^{\prime n}(p_{\bar{i}}) = x_{\bar{i}}^{\prime n}(p_{\bar{i}})$  for any  $k$  ex post, ex ante  $i$  does not know the value since it is dependent on the winner's (opponent's) bid price. However,  $i$  can determine the expected ex ante value of  $x_{i|\#(\bar{i})=k}^{\prime n}(p_{\bar{i}})$  when  $i$  bids  $p_i$  as

$$x_{i|k}^{\prime n}(\widehat{p_{\bar{i}}}) = \int_{\inf(S_{\text{IS}}^{1*})}^{p_i} \left[ \frac{(1-a)}{(1-a) + a(1 - F_{\text{IS}}^{\bar{i}1}(p))} \right]^n \frac{\{k f_{\text{IS}}^1(p)(1 - F_{\text{IS}}^1(p))^{k-1}\}}{1 - (1 - F_{\text{IS}}^{\bar{i}1}(p_i))^k} dp \quad (34)$$

The first multiplicative term represents the belief held in the second period by the first period winner which won the first period by bidding a price  $p$ . In the second term, the term in braces represents the probability distribution of the winning bid when the winner is not seller  $i$  and the

denominator conditions on having at least one of the  $k < n$  sellers in the reverse-auction. The limits of the integration are the possible winning price values that the opponent can bid and win, while seller  $i$  bids  $p_i$ . Using equations, 32, 33 and 34, we compute

$$\Pi_i(p_i) = \left[ (1-a) + a[1 - F_{\text{is}}^{\bar{i}1}(p_i)] \right]^n p_i + (1-a)^n - (1-a)^n \log \left( [(1-a) + a[1 - F_{\text{is}}^{\bar{i}1}(p_i)]]^n \right)$$

Further:

**Proposition 3.17**  $\sup(S_{\text{is}}^{1*}) = 1$ .

By definition, mixed strategy equilibrium generates the same profits for its opponents independent of any price bid the opponent i.e., it generates the same profit for  $i$  independent of whether  $i$  bids  $p = 1$  or any other price. Since  $F_{\text{is}}^1(1) = 1$  (from the last proposition),  $F_{\text{is}}^1(p_i)$  is a solution to the following

$$\begin{aligned} (1 - aF_{\text{is}}^{\bar{i}1}(p_i))^n p_i + (1-a)^n (1 - n \log(1 - aF_{\text{is}}^{\bar{i}1}(p_i))) &= \Pi_i(1) \\ &= 2(1-a)^n - n \log(1-a) \end{aligned} \quad (35)$$

Note that  $F_{\text{is}}^1(p)$  cannot be solved as a closed form expression. However, the equilibrium can be computed numerically for a given  $n$  and  $a$ . Although this limits us from obtaining an analytical expression for  $F_{\text{is}}^1(p_i)$ , one can further characterize the strategy set

**Proposition 3.18**

$$\inf(S_{\text{is}}^{1*}) = (1-a)^n (1 - n \log(1-a)) \geq (1-a)^n (1 + na)$$

**Proof:** Let  $\underline{p} = \inf(S_{\text{is}}^{1*})$ . To determine  $\underline{p}$ , we set  $F_{\text{is}}^1(\underline{p}) = 0$  in equation 35. This implies that

$$\underline{p} + (1 - a)^n = \Pi_i(1)$$

Rearranging the terms, we have

$$\underline{p} = (1 - a)^n + (1 - a)^n n (-\log(1 - a))$$

Using the power series expansion for  $\log(1 - a)$ , we can say that

$$\underline{p} \geq (1 - a)^n (1 + na)$$

**QED**

Based on this, we compute the expected price paid by the first buyer as

$$\hat{p}_{\text{is}}^1 = \sum_{k=1}^{n+1} \left[ (1 - a)^k a^{n-k+1} \binom{n+1}{k} \right] \left\{ \int_{(1-a)^n}^1 [k (1 - F_{\text{is}}^1(p))^{k-1} f_{\text{is}}^1(p)] p dp \right\} \quad (36)$$

In this expression, the term in the second square bracket represents the probability distribution of the minimum price observed when there are  $k$  sellers in the market. The first square brackets represent the probability of having  $k$  sellers in the market.

Having computed  $\hat{p}_{\text{is}}^1$  and  $\hat{p}_{\text{is}}^2$ , the total expected minimum price in the CIS is given by:

$$\hat{p}_{\text{is}} = \hat{p}_{\text{is}}^1 + \hat{p}_{\text{is}}^2 \quad (37)$$

## 4 Buyer Surplus Comparison

Since the higher the buyer surplus the lower the expected price, we compare only the expected price paid by the buyer. In order to show that the buyer surplus generated in IIS is lower than that in the CIS, we have to prove that

$$Diff = (\hat{p}_{\text{IIS}}^1 + \hat{p}_{\text{IIS}}^2) - (\hat{p}_{\text{CIS}}^1 + \hat{p}_{\text{CIS}}^2) > 0 \quad (38)$$

Note that  $p_{\text{IIS}}$  and  $p_{\text{CIS}}$  represent the expected price paid by the buyer in the IIS and the CIS settings.

The superscript represents the period.

Recall that the cdf of the bid distribution under IIS is intractable. Fortunately, we are still able to prove that  $Diff > 0$  assuming the best case scenario for IIS from a buyer surplus perspective.

Specifically, we assume the following for IIS:

1. In the first period, the lowest price observed is  $(1 - a)^n(1 + na)$ .
2. The winner always updates his belief such that  $x' = (1 - a)$ .
3. In the second period, the bidding behavior of the winner is similar to the loser.

Using the last two points, we have set  $\hat{p}_{\text{IIS}}^2 = \hat{p}_{\text{CIS}}^1$ . Therefore,

$$Diff = \hat{p}_{\text{IIS}}^1 - \hat{p}_{\text{CIS}}^2 \quad (39)$$

From the assumption in point 1 and the expression for  $\hat{p}_{\text{CIS}}^2$  from equation 10, we have the difference as

$$[(1 - a)^n + na(1 - a)^n] - [(n + 1)a(1 - a)^n] \quad (40)$$

Rearranging the terms, we have

$$Diff = (1 - a)^{n+1} \quad (41)$$

Since  $Diff > 0 \forall a \in [0, 1]$ , we have proved that the expected buyer surplus across both periods is lower in the *incomplete* information setting than in the complete information setting. But, our interest lies in studying which period contributes to this result.

## 4.1 Period by Period Comparison

Consider the expressions for the expected price paid by the first buyer under CIS and IIS settings. The expressions for  $\hat{p}_{CIS}^1$  is similar to  $\hat{p}_{IIS}^1$  except for the bid distribution. We demonstrate that the cdf of the first period bid distribution for IIS is first-degree stochastic dominant over CIS. Using that, we conclude that the first period buyer surplus is higher under CIS than under IIS.

In order to demonstrate the first-degree stochastic dominance, we compute the difference between IIS and CIS first period distributions as a function of the price bid,  $p$ . Recall that the first period bid distribution for IIS,  $F_{IIS}^1(p_i)$ , is a solution to the following equation

$$(1 - aF_{IIS}^1(p))^n p + (1 - a)^n (1 - n \log(1 - aF_{IIS}^1(p))) = 2(1 - a)^n - n \log(1 - a)$$

In this equation, if we replace  $F_{IIS}^1(p)$  with the expression for  $F_{CIS}^1(p)$  from equation 13, we obtain the difference between the two first period distributions. The difference is

$$Diff = -(1 - a)^n \log p$$

Note that this difference is zero only when  $p = 1$ . In other words, only when  $p = 1$  the two

distributions are equal. This also implies that for  $p < 1$ , one distribution is greater than the other and, for  $p > 1$ , vice versa. Our interest lies in characterizing this only for  $p < 1$ . From proposition 3.14 and proposition 3.18, we can state that  $F_{\text{CIS}}^1(p) > F_{\text{IIS}}^1(p)$ . Thus,  $F_{\text{IIS}}^1(p)$  is first degree stochastic dominant over  $F_{\text{CIS}}^1(p)$  and  $\hat{p}_{\text{IIS}}^1 > \hat{p}_{\text{CIS}}^1$ <sup>4</sup>. Stated otherwise, it is in the interest of the first buyer to choose CIS over IIS. Note that this result contradicts the traditional view that under uncertainty sellers are worse off and buyers better off. The key contribution of this paper is in emphasizing the importance of the nature of uncertainty to compare the performances of the frameworks with and without it.

## 5 Conclusion

In conclusion, we have addressed an important real-world problem faced by the buyer in a reverse-marketplace – the impact of market structure uncertainty on buyer surplus. The lack of prior work addressing this problem can be attributed to the novelty of the problem. Without the Internet or information technologies, such policies were not feasible and therefore, the need for such a study did not exist. However given the current scenario, this is an important IT-enabled problem. In line with this motivation, we analytically compare two of the many policies that are possible in e-marketplaces. We observe results that appear counter-intuitive: the setting with market structure uncertainty generates a lower buyer surplus than that without it. Insights gained explain why, in order to maximize buyer surplus, it may be best to choose a policy that generates the least level of market structure uncertainty for sellers. This actionable result was presented to Freemarkets. During that discussion, we learned that the seller behavior captured by our model is observed

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<sup>4</sup>One cannot categorically state if  $\hat{p}_{\text{IIS}}^2 > \hat{p}_{\text{CIS}}^2$  or not for a generic setting. But, we have proved that  $\hat{p}_{\text{IIS}}^2 > \hat{p}_{\text{CIS}}^2$  for  $n = 1$  i.e., 2 seller case in Arora *et al.* (2002).

during most reverse-auctions. The only scenario where the behavior differs is when sellers have the ability to collude.

In addition to aiding the buyer in choosing the appropriate revelation policy, our analysis also contributes to the understanding of the impact of different types of uncertainty on bidder behavior and buyer surplus. We contrast the difference between market-structure uncertainty and cost uncertainty. When uncertain about their own costs, sellers bid a lower price in order to “learn” their costs. This leads to a higher buyer surplus in the first period when compared to the setting without it. But with market structure uncertainty, it is the opposite. Only by bidding a high price, sellers can “learn” their monopolistic position. This leads to lower buyer surplus in the first period than in the setting without it.

We intend to extend this work by computationally comparing the two revelation policies studied in this paper. We expect to relax the following assumptions: a) In this paper, we set the participation correlation across the two periods to be one for tractability reasons. In reality, it is not so. We plan on comparing the policies for different exogenously set correlation values. b) Also for tractability reasons, we consider only a two period game in this paper. We propose to study the impact of relaxing this assumption on buyer surplus. c) Finally, we intend to compare the revelation policies in a framework where sellers have different cost-structures and where they face both types of uncertainties – market structure uncertainty and uncertainty about opponent’s cost-structure.

## References

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