Flexible Affordable Housing Policy Design with Scale Impacts and Equity Objectives

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Abstract:

Affordable and subsidized housing providers must design and implement housing strategies: where, when and with what types of housing to best meet the needs of low- and moderate-income households for affordable/low-cost permanent shelter. Recent research has proposed a multiobjective integer programming model for this purpose that jointly optimizes measures of net social benefit and equity and addresses limited variations in housing characteristics.

In this paper, we extend the affordable housing planning model to better reflect current research and practice in affordable housing and better meet the needs of affordable housing providers. First, we allow for scale impacts in net social benefits and provision costs that result in a discontinuous, piecewise-linear net social benefit objective. Second, we define alternative nonlinear social equity objectives that better capture social concerns associated with increased class and residential integration.

Computational results indicate that these affordable housing model variants are computationally tractable and provide substantial planning flexibility in both decision space and objective space. Planning models incorporating equity objectives that minimize the maximum negative impacts on potential destination communities and minimize variance in impacts on labor markets provide the greatest policy insights.

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Affordable housing, integer programming, multi-objective programming, urban affairs
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I. Introduction

Policy Motivation

Affordable housing in the United States is shelter whose mortgage or rent cost does not exceed 30% of household income, excluding units that are moderately or severely inadequate (U.S. Department of Housing and Urban Development 2001). While affordable housing is a concern of families of all incomes, much housing policy analysis addresses the availability of affordable housing for lower income families, i.e. those with incomes less than 80% of the area median income (AMI). In addition, policymakers focus on reducing the number of families with “worst-case” housing needs, defined as very-low-income families (those with incomes less than 30% of AMI) who spend more than half of their income on housing or who live in severely inadequate housing (U.S. Department of Housing and Urban Development 2001). Recent studies have identified a shortage of good-quality affordable housing for low- and moderate-income families in the U.S. Estimated deficits range from 1.8 million units (Millennial Housing Commission 2002) to 4.4 million units (Center for Budget and Policy Priorities 1998).

Affordable housing may be provided in a variety of ways, including housing subsidies, court-mandated set-asides, inclusionary zoning, “smart growth” initiatives and affirmative marketing efforts. Despite the best efforts of organizations across the government, for-profit and nonprofit sectors, there remain structural impediments to meeting the housing needs of poor and working families. Many affluent communities, looking to maximize their fiscal well-being and quality of life, enforce land-use policies that limit access of lower-income families to affordable housing. Larger urban areas, seeking to stabilize their tax base, are forced to make difficult tradeoffs between attracting and retaining affluent families and meeting basic needs of lower-income families. Many affordable housing providers do not have the resources to estimate the social benefits and costs of alternative affordable housing provision strategies.

Quantitative planning models for the provision of affordable housing have the potential to generate significant social returns similar to those in areas such as public safety, transportation and environmental planning. The goal of this paper is to address limitations in existing planning models for
affordable housing provision, and to provide preliminary computational results for the more complex yet realistic planning models that result.

*Previous Work*

The existing literature on prescriptive planning models for affordable/subsidized housing provision is small. Kaplan (1986) has developed models for the design of waiting lists for existing and newly-renovated public housing. Forgionne (1991) applies operations research methods to assist the U.S. Army in forecasting demand for military housing, and to allocate resources to secure such housing. Johnson and Hurter (2000) present a multiobjective optimization model for housing policy design inspired by housing mobility programs such as the Gautreaux Assisted Housing Program. Extensions include a simplified model for tenant-based subsidized housing using a larger and more realistic dataset (Johnson 2003) and new model for project-based subsidized housing (Johnson 2006b).

However, these recent research results are limited in scope and applicability. Subsidized housing, defined as housing for low- and moderate-income families provided below cost and financed primarily by the Federal government, accounts for only 28.8 percent of all non-military Federal support for housing (Millenial Housing Commission 2002, p. 15). Thus, research in housing planning ought to address the larger universe of affordable housing provided by actors across multiple sectors.

Johnson (2006a) developed a comprehensive framework for affordable housing location using two complementary planning models: a *strategic* model intended to identify broad investment priorities, and a *tactical* model intended to identify specific location schemes. The tactical model has limited descriptive validity: it assumes no scale economies in dollar-valued impacts of affordable housing and provides little empirical evidence to support its equity objective. We extend this initial effort to provide substantive policy and managerial insight to the theory and practice of affordable housing location.

*Summary of Findings*

Affordable housing practice and research provides evidence of scale economies and
diseconomies in affordable housing provision. Also, application of results from housing and urban policy research yields two equity objectives not previously used for affordable/subsidized housing location models. The basic affordable housing planning model is modified to reflect a wider range of housing characteristics and equity objectives. Computational results using commercially-available modeling software and solver have descriptive validity and show variation in objective and decision space that provide flexibility to decisionmakers regarding housing strategy. In particular, in spite of challenging functional characteristics of a multiple objective nonlinear integer program including an employment-related equity objective, compromise nondominated solutions associated with this model compare favorably with those generated using a minimax center objective used in previous papers. Finally, a number of extensions of and variations on the basic affordable housing planning problem provide the basis for a research agenda that is challenging and relevant to practitioners.

Paper Organization

Section II reviews affordable housing characteristics, and justifies the use of the fixed-charge location problem as the primary analytical tool. Section III presents a variation of the affordable housing location model of Johnson (2006a) that incorporates scale economies in fixed provision costs and variable net social benefits. Section IV provides formulations for two new equity measures justified on the basis of location theory and social need. Section V contains computational results. Section VI discusses policy implications and describes model extensions. Section VII concludes.

II. Housing as a Location Problem

Descriptive and Normative Analysis of Housing Planning

The problem of market-rate residential housing location has traditionally been approached using two very different methods. Urban planners and regional scientists have since the late 1960s developed housing market equilibrium models that calculate residential and worker population distributions as a function of housing, employment and transport characteristics. Such models may be modified by
incorporating the choice of transport mode, and by addressing the multi-period nature of actual regional economies (see e.g. Anas and Arnott 1994). These models are of most interest to researchers seeking to understand the underlying dynamics of simultaneous housing, employment and transport choice.

Real estate practitioners, on the other hand, generally apply a great deal of practical knowledge to this problem, in particular awareness of regional employment growth trends, demand for various types of housing, zoning and land use restrictions, engineering and construction and housing finance. With business analytical support, practitioners often perform quick and informal analyses to select a few candidate sites and choose the type and mix of housing likely to be financially viable. The remainder of their work addresses negotiations with various stakeholders to fund and build the development (Miles, Berens and Weiss 2000).

Since the mid-1990s, the U.S. residential housing market has enjoyed a rise in new home construction, existing home sales and home values (Green and Malpezzi 2003). Moreover, American consumers have asserted, through purchases and survey attitudes, their preferences for single-family homes on separate lots, housing types more often found in the suburbs than the central city (Carliner 2001). However, there is a small but increasing fraction of consumers who desire denser housing consistent with New Urbanist values (Myers and Gearin 2001). There appears to be a conflict between policy prescriptions by housing researchers and preferences of consumers and housing providers.

Location of subsidized housing in the US has historically been performed using a variant of the real estate practitioners’ approach. That is, public housing authorities (PHAs) selected sites, designed developments and built and managed housing, but under the additional constraint that such housing be located, for the most part, in areas where poor families already live or on least-valuable land in the metropolitan area. The long-term negative impacts, both on these public housing communities and on the financial resources that support them, of this type of subsidized housing location have been well known for over thirty years (Rubinowitz and Rosenbaum 2000).
In reaction, two primary government policy initiatives have been developed. The first, tenant-based housing subsidies, commonly known as the “Section 8” program, has since the mid-1970s enabled families to choose market-rate rental housing throughout a metropolitan area, using rental subsidies provided directly to the landlord by the PHA. In practice, families with Section 8 subsidies face significantly reduced housing options as compared with more affluent families, but have generally managed to choose neighborhoods that are more advantaged than the very-low-income communities in which public housing is concentrated (Turner, Popkin and Cunningham 1999). The second, mixed-income project-based housing, requires PHAs, alone and in partnership with not-for-profit organizations, to directly provide new housing to tenants of varying economic backgrounds. In addition, this housing is intended to be either (a) located in more affluent communities than those in which public housing has traditionally been located or (b) located in low-income communities undergoing intensive economic revitalization (Popkin et al. 2004). High levels of racial residential and housing type segregation in the United States and challenges facing affordable housing providers are indications that significant inefficiencies remain in the design and implementation of subsidized housing policy.

The limited research on prescriptive models for subsidized and affordable housing design has focused on three complementary areas. The first is stylized policy models that allow estimation of long-term population impacts of large-scale initiatives such as housing mobility programs (see, e.g. Caulkins et al. 2005). The second area is more detailed math programming models that recommend specific allocations of various types of assisted housing: tenant-based subsidized housing (Johnson 2003), project-based subsidized housing (Johnson 2006b) or affordable housing (Johnson 2006a). The last area of prescriptive modeling is decision support models and applications that enable individual families to choose neighborhoods and specific housing units (Johnson 2005). The goal of this paper is to extend math programming-based housing planning models to incorporate more aspects of real-world development.

*How Does Housing Differ from Other Facilities?*
There is limited understanding of housing as a subject of facility location models. Johnson (2006b) argues that subsidized housing differs from traditional facilities such as warehouses, stores and service facilities in the nature of physical proximity between customers and units and the type and magnitude of externalities generated. For example, proximity of families to housing is not a primary consideration of its desirability; families may search for housing in their local neighborhood, across the city or even across metropolitan areas. Also, occupants of assisted housing enjoy greater locational and financial stability and improvements in neighborhood amenities such as school quality and local services.

Affordable housing may be sited close to residents who may be public transit-dependent. But it may also be sited far away from centers of low- and moderate-income residents, both to meet policy concerns of poverty deconcentration and class integration, and to enable low and moderate-income households to live close to their workplaces. Thus, traditional modeling assumptions such as average distance (see e.g. Hakimi 1964) or coverage (see e.g. Church and ReVelle 1974) as a primary metric of utility are insufficient.

Finally, affordable housing and especially subsidized housing may be seen by potential neighbors as a source of disamenities such as increased criminal offending and reduced property values. Thus, one may view affordable/subsidized housing as an “undesirable” good to be dispersed as much as possible across a study area and away from vulnerable communities. However, housing is intended to be located in a residential context. Since many residential neighborhoods have amenities that are desirable to a wide range of families, “dispersion” facility location models and variants (see e.g. Kuby 1987) do not appear appropriate for our purposes.

*The Fixed-Charge Location Problem as a Basis for Affordable Housing Design*

While housing is obviously “sited”, practitioners who do so tend not to rely on the traditional slate of facility location models. Yet it is asserted here that prescriptive planning models for housing have the potential to add value to the work of practitioners and researchers. How might this be done?
In this paper we adapt the fixed-charge facility location model (Balinski 1964), in which facilities are sited to serve spatially dispersed customers to minimize the sum of fixed location costs and variable distribution costs, subject to a budget constraint. This model seems particularly useful here since housing is a resource-intensive good for which fixed costs, represented by site acquisition and construction costs are just as relevant for housing as for other facilities such as warehouses or libraries. We re-frame variable costs as benefits that accrue to families in housing and costs to society that vary in the number of units sited in a neighborhood, independent of distance measures.

III. Affordable Housing Location Model with Scale Effects

Nature of Scale Economies

There is a wide variety of attributes by which an affordable housing initiative may be classified (Johnson 2006a). These attributes apply to particular entities associated with the initiative: housing unit, housing structure, containing one or more units, housing development, consisting of one or more structures, and housing project, comprising one or more developments. For example, a housing unit may be characterized by tenure type, subsidy type, and size; a housing structure may be characterized by provision method, residential type, size and subsidy type; a development may be characterized by contiguity, income diversity and development time, and a project may be characterized by unique attributes of structures and developments over a given development horizon. In addition, we assume that each unit is occupied by a single household which enjoys economic benefits associated with good-quality affordable housing.

Aboolian (2002) formulates and solves the location-design problem of simultaneously choosing values for a wide range of facility attributes, and choosing a location for the facility. We adapt Aboolian’s conceptual framework to standard housing development practice and determine whether each of a wide variety of housing attributes is more likely to be associated with fixed provision costs or with variable net social benefits. We classify an attribute as quantititative or qualititative in nature, and determine whether its
range of values demonstrates scale economies or diseconomies (if quantitative), or whether its values are likely to contribute to scale economies or diseconomies (if qualitative). Table 1 contains the details:

[Table 1: Analysis of Housing Initiative Attributes]

Note that some attributes apply to both housing structures and housing units, such as subsidy and size. This analysis indicates a potentially wide range of choices available to housing modelers.

Initially, we focus on fixed costs and variable social impacts solely as a function of the number of housing units in a development. Fixed costs of housing development are divided into two categories: hard costs, including site acquisition, construction and site improvement, and soft costs, including engineering, design, legal and developer fees (Miles, Berens and Weiss 2000). Real estate development principles place significant emphasis on correctly estimating these costs in order to ensure that “sources” of development funds match “uses”. However, research analysis on the characteristics of fixed costs as a function of housing development characteristics is limited.

It is likely that fixed costs tend to increase in the size of the development but at a decreasing rate, due to discounts on prices of supplies and professional skills applied to multiple units simultaneously. An analysis of fixed costs associated with seven projects produced by an actual affordable housing developer (details available from author) indicates that total hard costs and total fixed costs increase at a decreasing rate as a function of the number of units in a structure or the number of structures in a development.

There is a wide range of social impacts associated with subsidized housing, including participant impacts (e.g. housing consumption, life outcomes) and non-participant impacts (property value, crime rates, subsidy levels). All of these impacts can be measured in dollars using social benefit-cost analysis (Johnson, Ladd and Ludwig 2002). There are few studies that have focused on social impacts as a function of the size of the development in which families live, as opposed to demographic characteristics.
of destination neighborhood\(^1\). Given a certain level of low-income families in a housing development, it seems reasonable to assume that social benefits to low-income families might increase initially as the size of the development and number of low-income families increases, as a result of increased social networks enabling families to find employment, child care, companionship and other necessities. However, social benefits might subsequently show decreasing returns to scale due to concentrations of socio-economic disadvantage and difficulties in maintaining the physical quality of the development.

Scale economies can be incorporated into facility location models by discretizing impacts (see e.g. Osleeb \textit{et al.} 1986). Let \(x\) denote the size of a housing development. Suppose that there are three size classes of housing developments, indexed by \(j = 1, 2\) and \(3\). “Small” housing developments consist of those in which \(x \leq l_1\) units; “medium” housing developments consist of those in which \(x \in (l_1, l_2]\); “large” housing developments consist of those in which \(x \in (l_2, l_3]\). Denote the fixed provision cost of a size-\(j\) project in a typical neighborhood as \(f_j, j = 1, 2, 3\). Denote the per-family net social benefit of a size-\(j\) project in a typical neighborhood as \(b_j, j = 1, 2, 3\). The \(f_j\) are interpreted as the average height of a provision cost curve for \(x \in (l_{j-1}, l_j)\); scale economies for provision costs are represented by the relationships \(f_{j+1} \leq f_j\) and \(\frac{(f(l_j) - f(l_{j+1}))}{(l_j - l_{j+1})} \geq \frac{(f(l_{j+1}) - f(l_j))}{(l_{j+1} - l_j)}\). The \(b_j\) are interpreted as the slope of a net social benefits curve at \(x \in (l_{j-1}, l_j)\), and decreasing scale economies mean that \(b_{j+1} \geq b_j\). For simplicity we interpret \(l_0 \equiv 0\) and set the breakpoints \(l_j\) to be identical net social benefits and provision costs. We illustrate dollar-valued impacts a housing development of various sizes in Figure 1, below:

[Figure 1: Net Benefits of a Housing Development with Scale Effects]

\(^1\) Galster (2002) has explored a variety of models of social impacts of housing deconcentration as a function of the level of concentration of low-income families across neighborhoods. He concludes that net social benefits may show scale economies as families relocate from moderate-poverty neighborhoods to low-poverty neighborhoods, but net social benefits show scale diseconomies as families relocate from high-poverty neighborhoods to moderate-poverty neighborhoods.
Notice that the total cost function is discontinuous and piecewise linear.

**Affordable Housing Location Problem with Scale Effects**

Johnson (2006a) presents a multi-objective integer program for affordable housing provision that distinguishes between housing tenure types but assumes no scale economies or diseconomies in social impacts or provision costs. That model recognizes that affordable housing provision can be controversial and uses the center, or minimax equity measure to minimize the perceived negative impacts of affordable housing on most-affected neighborhoods. We modify that formulation to incorporate scale effects, and for simplicity focus on a single housing tenure type. As in Johnson (2006a), we assume that a strategic affordable housing planning problem has generated an estimate of the total number of units of a particular housing tenure type to locate.

**Indexes and Sets:**

\[ i = 1, 2, \ldots, n \]: Candidate neighborhoods for housing developments; \( I = \{i\} \)

\[ j = 1, 2, \ldots, m \]: Housing development size categories; \( J = \{j\} \)

**Data:**

\[ E_i: \] Excess of demand for affordable housing over supply in neighborhood \( i \);

\[ b_{ij}: \] Per-unit net social benefit for a size-\( j \) development in neighborhood \( i \);

\[ f_{ij}: \] Fixed provision cost for a size-\( j \) development in neighborhood \( i \);

\[ l_i: \] Lower bound on number of units in a size-\( j \) development;

\[ F: \] Total number of units to be developed in a planning period;

\[ M: \] Large number
Decision Variables:

\[ x_{ij} = \begin{cases} 
1, & \text{if a size-}j \text{ housing development is located in neighborhood } i \\
0, & \text{otherwise}; 
\end{cases} \]

\[ z_{ij} = \text{number of units built in a size-}j \text{ development located in neighborhood } i \]

Model (AFFLOC):

Maximize

\[ \sum_i \sum_j b_{ij} z_{ij} - \sum_i \sum_j f_j x_{ij} \]  

(1)

Minimize

\[ \text{Maximize} \quad \frac{\sum_{j=1}^n z_{ij}}{F} \]

\[ \text{subject to:} \]

\[ 0 \leq z_{ij} \leq I_j - l_{j-1} \quad \forall i, j \]  

(3)

\[ z_{ij} \leq M \cdot x_{ij} \quad \forall i, j \]  

(4)

\[ \sum_{j=1}^m z_{ij} \leq E_i \quad \forall i \]  

(5)

\[ \sum_{i=1}^n \sum_{j=1}^m z_{ij} = F \]  

(6)

\[ x_{ij} \text{ binary} \quad \forall i, j \]  

(7)

\[ z_{ij} \text{ integral and non-negative} \quad \forall i, j \]  

(8)

Objective (1) maximizes the difference between net social benefits, which show scale diseconomies, and provision costs, which show scale economies. As shown in Figure (1), this is a piecewise linear and
discontinuous function. Objective (2) minimizes the maximum disparity across neighborhoods between
the fraction of all affordable housing to be sited which is allocated to a single neighborhood and that
neighborhood’s need for affordable housing.

Constraints (3) ensure that the number of housing units in a size-\(j\) development is consistent with
definitions of size categories. Constraints (4) enforce that the number of units in a size-\(j\) development in
neighborhood \(i\) can be nonzero only if a size-\(j\) development is actually sited in neighborhood \(i\).
Constraints (5) limit the number of units located in neighborhood \(i\) to be less than or equal to that
neighborhood’s affordable housing gap. Constraint (6) sets the total number of housing units sited across
all neighborhoods to be exactly \(F\). Constraints (7) and (8) ensure that location decision variables are
binary, and size decision variables are integral and non-negative.

Discussion

Let \(n = |I|\) and \(m = |J|\). The affordable housing location problem with scale effects (AFFLOC) has
\(mn\) binary variables, \(mn\) general integer variables and \(mn + n + 1\) constraints. In practice, it is likely that
\(n\) is much larger than \(m\), and that \(m\) is o(1), so that this problem is o(n) in decision variables and
constraints. The single-objective version (1), (3) – (8) is similar to a capacitated fixed-charge location
problem with piecewise-linear discontinuous objective. Though there are published algorithms on solving
continuous (Fourer 1992) and discontinuous (Conn and Mongeau 1998) piecewise-linear math programs,
there are no algorithms known to this author that address discontinuous piecewise-linear integer
programs. Incorporating equity objective (2) results in a nonlinear discontinuous multiobjective integer
program which is even more challenging to solve. In Section V we solve AFFLOC by representing
objective (1) using special features of a math modeling language and replacing objective (2) with a linear
equivalent.

IV. Social Equity Measures

Analyzing Equity Measures for Location Models
Affordable housing location implies social impacts, real or perceived, that are not easily monetized and thus must be represented as “equity” or fairness objectives that serve as a counterbalance to efficiency objectives. However, the equity objective (2) in Section III only addresses potential political opposition to affordable housing, and does so using a “center”-type measure chosen primarily for analytic tractability. In this section we generate alternative equity measures using the analytical framework of Marsh and Schilling (1994).

*Equity Measure: Spatial Segregation of Affordable Housing*

One promising equity measure to address political opposition to housing mobility is the *index of dissimilarity (DI)* (James and Tauber 1985). DI measures the variability of one quantity relative to another quantity across neighborhoods. It has been commonly used to measure the level of racial segregation. In the context of affordable housing planning, DI may compare the number of families in assisted housing in each neighborhood to that neighborhood’s total population. A value of DI equal to zero means that the fraction of each neighborhood’s population that lives in assisted housing is equal to the fraction of the population of the entire study area that lives in assisted housing; a value of DI equal to one means that assisted housing is entirely segregated. Assume that lower values of DI are more desirable on social welfare grounds, and perhaps less likely to result in political opposition from aggrieved destination communities than are larger values. Then we may replace the “center” equity objective (2) by an adaptation of DI. Below we show how this can be done.

Given an allocation \( z \) of families in assisted housing about neighborhoods \( i \) of the study area, each with \( n_i \) families, \( a_i \) of which live in assisted housing define \( DI(z) \) as:

\[
DI(z) = \sum_{i=1}^{m} \frac{\bar{n}_i \ | p_i - P |}{2NP(1-P)}, \text{ where:}
\]

\[
\bar{n}_i = n_i + \sum_{j=1}^{m} z_{ij} \text{ is the total number of families in neighborhood } i;
\]
\[ p_i = \frac{a_i + \sum_{j=1}^{m} z_{ij}}{n_i + \sum_{j=1}^{m} z_{ij}} \] is the fraction of all families in assisted housing in neighborhood \( i \);

\[ N = \sum_{i=1}^{n} \bar{N_i} \] is the total number of families of the study area, and

\[ P = \frac{1}{N} \sum_{i=1}^{n} \left( a_i + \sum_{j=1}^{m} z_{ij} \right) \] is the fraction of all families that live in assisted housing.

Using the classifications of Marsh and Schilling (1994), \( DI \) is normalized, obeys the principle of transfers and in addition is easily interpreted. While not scale invariant, \( DI \) does generate meaningful dispersion values even for very small numbers of affordable housing program participant households in each neighborhood. Also, though \( DI \) is not Pareto-optimal, it is used in a multiobjective program designed to generate Pareto-optimal solutions. \( DI \) has not to our knowledge been used in a published operations research-based planning model.

**Equity Measure: Proximity to Employment Opportunities**

A key policy motivation for an employment-related equity measure is the existence and extent of a spatial mismatch of workers in lower-income and predominately-minority areas, often in central cities, and employment opportunities in more affluent, higher job-growth areas. There is widespread, though not unambiguous support for this “spatial mismatch hypothesis” (Ihlandfeldt 1999). Computing an equity measure capturing elements of spatial mismatch requires data on distances between locations of assisted housing families to local employment centers. Since our affordable housing model does not rely on distances between “markets” and “customers”, we rely on the “relative burden” criterion used in the minimax equity measure (eq. (2)).

We use this measurement in a variance equity framework because we wish to strongly penalize deviations from the mean value. Equivalently, we want to ensure that families allocated to different regions have similar employment opportunities. We make the partial-equilibrium modeling assumption
that labor markets are not significantly affected by increases in the number of assisted housing families in certain areas, a reasonable assumption if the scale of the assisted housing program under consideration is not large. This variance equity measure is defined as:

\[
E(v_1, v_2, \ldots, v_m) = \sum_{l=1}^{R} \left( \frac{v_l}{J_l} \right) \left( \frac{1}{R} \sum_{m=1}^{g} \frac{v_m}{J_m} \right)^2 ,
\]

(10)

\(l(m)\) = 1, 2, \ldots, \(R\) is an index of employment submarkets;

\(S_l\) = the set of indices of neighborhoods contained in employment submarket \(l\)

\(v_l = \sum_{i \in S_l} \sum_{j=1}^{m} z_{ij}\) is the number of affordable housing units sited in employment submarket \(l\);

\(J_l\) = number of entry-level jobs in employment submarket \(l\);

\(J = \sum_{l=1}^{R} J_l\) is the total number of entry-level jobs in the study area;

and other quantities are as previously defined.

The variance measure (10) obeys the principle of transfers and is scale-invariant, but is not Pareto-optimal and is not normalized. This measure, though requiring specialized optimization techniques, has been used in a number of public-sector OR/MS models (see, e.g. Berman 1990).

We report computational results for model AFFLOC with equity measures (9) and (10) replacing the center measure (2) below.

V. Computational Results

Model Analysis and Reformulations

We linearize the “center” equity objective (2) through a common linear programming reformulation:
Minimize $D$ \hfill (11a)

Subject to:

\[
D \geq \begin{cases} 
\sum_{j=1}^{m} \frac{z_{ij}}{P} & \frac{E_{i}}{\sum_{j=1}^{n} E_{j}} \quad \forall i \in I, E_{i} > 0
\end{cases}
\] \hfill (11b)

The segregation equity index (9) can be linearized as well. Since $\sum_{i=1}^{n} \sum_{j=1}^{m} z_{ij} = F$, $Dl(z)$ may be expressed as a function of nonnegative, continuous decision variables $d_{i}^{+}$ and $d_{i}^{-}$:

\[Dl(z) = C \cdot \sum_{i=1}^{n} (d_{i}^{+} + d_{i}^{-})\], where:

\[C = \frac{1}{2 \cdot (A_{o} + F) \cdot \left( \frac{N_{o} - A_{o}}{N_{o} + F} \right)}\] \hfill (13)

\[d_{i}^{+} - d_{i}^{-} = \left( a_{i} - n_{i} \cdot \left( \frac{A_{o} + F}{N_{o} + F} \right) \right) + \left( 1 - \frac{A_{o} + F}{N_{o} + F} \right) \left( \sum_{j=1}^{m} z_{ij} \right)\] \hfill (14)

\[d_{i}^{+} + d_{i}^{-} = \left( l_{i} - n_{i} \cdot \left( \frac{A_{o} + F}{N_{o} + F} \right) \right) + \left( 1 - \frac{A_{o} + F}{N_{o} + F} \right) \left( \sum_{j=1}^{m} z_{ij} \right)\] \hfill (15)

and $A_{o} = \sum_{i=1}^{n} a_{i}$ and $N_{o} = \sum_{i=1}^{n} n_{i}$.

The linear programming relaxation of employment equity objective (10) can be shown to be equivalent to the objective function of a quadratic program (Bazaraa, Sherali and Shetty 1993). By imposing the additional constraint that $v_{i}$ is discrete, (10) is now the objective function of a mixed-integer quadratic program. However, a number of solvers have implemented high-quality branch-and-cut solution algorithms for such problems.
Data

Data for model AFFLOC is derived from that used in Johnson (2006a). Estimates of fixed provision costs, subsidy levels and household-level housing expenditures are derived from administrative data provided by a Pittsburgh-area non-profit affordable housing provider. Average total development costs are $102,022 for rental units provided through rehabilitation, $149,679 for owner-occupied units provided through new construction and $107,040 for owner-occupied units provided through rehabilitation. Newly-constructed owner-occupied housing carries about $100,000 in subsidies per unit and nearly $900,000 per development, significantly more than other types of housing developed by the provider. This is consistent with typical developer experience in the Pittsburgh region; new housing construction on brownfield sites, as compared to rehabilitation of existing units, often entails costly demolition of existing structures and environmental remediation.

Estimates of family benefits using detailed economic models as in Johnson (2006b) are currently unavailable due to a lack of observations on households served by this provider. Nevertheless, as in Johnson (2006a), we rely on economic theory that says that because assisted or subsidized housing is an “in-kind” contribution (as opposed to a cash transfer), it is inherently economically inefficient. Thus, the subsidy associated with any assisted housing unit is an upper bound to the short-term benefits enjoyed by the occupants; these benefits are conventionally modeled as consumer surplus.

These data are adapted to the affordable housing location model with scale effects in the following way. All data used for the model instances solved in this section are derived from multifamily rental housing. Neighborhoods are represented as Census tracts. Actual values for tract-level dollar-valued family benefits, subsidies and provision costs are derived from point estimates based on administrative data and simulated using a uniform distribution with minimum and maximum values. “Small” housing projects are defined as those with 1 – 2 units; “medium” housing projects are defined as those with 3 - 5 units, and “large” housing projects are defined as those with 6 - 9 units. Fixed costs, subsidies and variable net benefits are assumed to apply to “medium”-sized rental housing developments.
Unit-level variable net benefits for “small” (“large”) projects are assumed to be 20% more (10% less) than corresponding values for “medium” projects. Fixed costs for “small” (“large”) projects are assumed to be 30% less (20% more) than the corresponding values for “medium” developments.

Estimates of gaps between demand and supply for affordable housing are adapted from Johnson (2006a). These values are derived from the American Housing Survey (AHS; U.S. Census Bureau 1995), which reports results according to “zones” which are aggregations of Census tracts. Johnson (2006a) found that affordable housing demand substantially exceeds supply for owner- and renter-occupied units in nearly every zone within Allegheny County.

The number of families in each Census tract, used in the segregation index, came from the 2000 Census (U.S. Census Bureau 2004). The number of assisted housing units in each Census tract was approximated by the number of tenant-based subsidized housing units in each tract, originally provided by local PHAs for use in a housing counseling decision support system (Johnson 2005).

The number of entry-level jobs in each Census tract was estimated using tabulation PCT 86 of the 2000 Census (“Sex by Occupation for the Employed Civilian Population 16 Years and Over”) and summing employment levels for Census employment codes representing unskilled or entry-level jobs. These counts were then aggregated to the AHS “zone” level, representing employment submarkets.

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2 AMPL’s piecewise-linear modeling facility << >> requires that functions be expressed as slopes of lines between breakpoints. Therefore, fixed-cost multipliers and breakpoints were chosen so that the slopes (average values) of fixed cost line segments were monotonically decreasing.

3 For example, under “Education, training and library occupations”, codes for secondary school teachers were excluded, while codes for instructors and “other teachers” were not. A more accurate measure of entry-level employment is contained in the Census Transportation Planning Package, a special Census tabulation prepared by the U.S. Department of Transportation, but these data use an areal unit of analysis incompatible with Census tracts.
We adapt results from Johnson (2006a) in which a “strategic” knapsack-like model applied to data for Allegheny County yields an estimate of 44 units of renter-occupied units to be developed using a subsidy of about $3.6 million.

Results

We implemented model AFFLOC using the mimimax equity objective with the AMPL algebraic mathematical modeling language (ILOG-CPLEX Division 2003a) and solved it using CPLEX 8.0 (ILOG-CPLEX Division 2003b). We used the noninferior set estimating method as presented in (Daskin 1995) to generate non-dominated solutions. After AMPL’s presolve step, each subproblem resulted in 533 binary variables, 533 integer variables, 1600 linear variables as well as 1077 constraints. Solution times ranged from 1 second (no weight on the modified center objective (11a)) to 12,105 seconds (weight of 10-7 on the net benefit objective (1) and weight of 1 on the center objective).

We discuss run results in objective function-space and decision-variable space. Figure 2 represents the approximation to the Pareto frontier generated by the planning model.

[Figure 2: Pareto Frontier, Model AFFLOC with Center Equity Measure]

The coefficient of variation associated with the two objective function values across five nondominated solutions is 32.70% and 42.35%, respectively. A candidate compromise solution associated with weights \((1.1 \times 10^{-6}, 1)\) has an equity measure of 1.44 that is 29% higher than that of the equity-maximizing solution and a net benefit measure of -$1.86 million that is 21.1% more negative than that of the net benefit-maximizing solution. The reasonably large range of objective function values, and the relatively small tradeoffs associated with the compromise solution suggest that the compromise solution has policy importance.

Figure 3 contains details of the distribution of the number of housing units in each Census tract according to the three size categories described above. Figure 4 shows the spatial distribution of the two corner solutions and the compromise solution.
These figures demonstrate that there is significant variation in housing project sizes and spatial distribution across the corner solutions and the compromise solution. This is an indication that our affordable housing planning model generates alternative allocations that provide design flexibility for housing developers.

Solving the AFFLOC with the segregation index (9) requires substitution of linear transformation (16) of the segregation index for the center measure (2), and the addition of equation (14) to the original set of constraints. Nondominated solutions derived as above required negligible processing time. Objective-space results (Figure 5) indicate an approximation to the Pareto frontier in which net benefit and equity measures vary much less than those associated with the solution using the center measure: the coefficient of variation for the net social benefit and segregation index objectives is just 0.12% and 0.33%, respectively. This result arises from the very small change in total assisted housing units in any Census tract as compared to the total number of housing units. Thus, there is no obvious candidate for a compromise solution.

There is also negligible variation in the distribution of housing development sizes across the nondominated solutions: all housing is provided in large-size developments. However, GIS analysis indicates significantly greater spatial dispersion among the nondominated solutions using the segregation index than for the center measure. This may result from a wide range of values of population and affordable housing at the Census tract level, as opposed to a smaller variation of values of population at the American Housing Survey-defined “zone” level used in the center objective. Overall, model AFFLOC with the segregation index provides relatively little policy insight with these small problem instances.
Using the employment equity objective (10) in AFFLOC results in a multiple objective nonlinear integer program (MONLP). MONLP are generalizations of nonlinear (continuous) programs, integer programs and multiple-objective linear programs and are usually very difficult to solve. Though the majority of the research literature on MONLP addresses interactive methods to approximate the Pareto frontier (see e.g. Vassilev, Narula and Gouljashki 2001), we argue that in a relatively new decision modeling domain such as affordable housing planning, the high level of imprecision in the configuration of a specific solution strategy justify non-interactive (a posteriori) methods. Once a Pareto frontier has been approximated, another, possibly multi-stakeholder decision process might address the choice of a most-preferred policy alternative.

Moreover, since objective (1) is linear and objective (10) has been shown to be quadratic, this version of AFFLOC is a multiple objective quadratic-linear program (MOQLP; Rhode and Weber 1981), for which optimal solutions to the single-objective weighted version of MOQLP are efficient, i.e. lie on the Pareto frontier. We thus apply the non-inferior set estimation method to AFFLOC as a MOQLP, using AMPL and CPLEX as above. All nondominated solutions, except for the one maximizing net benefit, required CPLEX processing times equal to the user-defined limit of 4 hours. The approximation to the Pareto frontier for this model is shown in Figure 6.

[Figure 6: Pareto Frontier, Model AFFLOC with Employment Variance Equity Measure]

While the coefficient of variation for net benefits, 34.72%, is similar to the value associated with nondominated solutions using the center objective, the coefficient of variation for employment variation, 123.04% far exceeds that for any other equity measure used in this study. We identify two similar candidate compromise solutions: (net benefits, employment variance) = (-$1.54 million, 6.0) and (-$1.53 million, 7.13). The percent achievement of most-desirable equity values for these candidate compromise solutions is 78.68% and 74.63%, respectively, and the percent achievement of most-desirable net benefit values is 99.28% and 99.92%, respectively. These results illustrate the skewness of the tradeoff curve and
indicate that these compromise solutions may come very close to optimizing a decisionmaker’s utility function.

The distribution of development sizes according to nondominated solutions shows significant variation, similar to the original formulation of model AFFLOC: the most equitable solution has the largest variation, and the most efficient solution concentrates units in large developments (Figure 7).

[Figure 7: Development Sizes, Model AFFLOC with Employment Variance Equity Measure]

Spatial analysis of the nondominated solutions in this case shows that the solution maximizing net benefits results in high spatial concentration of units, even in regions with low levels of entry-level employment, while solutions with more favorable equity measures distribute units across a wider number of regions with a bias towards those with higher levels of entry-level jobs.

VI. Discussion

Model Limitations and Policy Implications

The computational results presented here are preliminary in a number of respects. First, Census tract-level dollar-valued impacts of affordable housing project impacts are derived from simulations based on point estimates rather than a forecasting model based on economic principles. This limits the policy insight to be derived from particular solutions to AFFLOC. Second, the length of CPLEX run times, especially for the multiple objective quadratic-linear program, indicate the need for more efficient solution methods that may provide better approximations to the Pareto frontier.

Nevertheless, multiple, non-dominated solutions for the affordable housing location model with scale effects (AFFLOC) and alternative equity measures provide nontrivial policy insight. We start with results for the location model using the “center” equity measure. Since the compromise solution and the benefit-maximizing solution are similar according to housing unit sizes, efficiency objective values and spatial distribution, decisionmakers may have the flexibility to choose between these two solutions according to other considerations, such as site availability and provider resources. On the other hand, if
political opposition to affordable housing provision is a significant impediment to provider operations, the equity-maximizing solution may provide for flexibility in the size and timing sequence of developments.

Results for AFFLOC with the segregation index are less promising; there is little variation in either of the objective function values or in the distribution of development sizes across nondominated solutions. This model likely provides little specific guidance to decisionmakers unless the number of units to be sited increases substantially. However, such an increase in the scale of the problem might violate the partial-equilibrium assumption that housing markets are largely unaffected by location decisions prescribed by the model.

Finally, model AFFLOC with the employment variance equity measure offers an intriguing tradeoff between computation times, which are unfavorable, and more favorable objective-space and decision-space results. The two candidate compromise solutions have high, virtually indistinguishable achievement levels for both objectives, and one of them offers modest variation in development size. In addition, this model provides support for housing policy based on access to employment opportunities, which may be a more politically palatable argument than either housing integration or population-based housing “burden”.

Results for all three models support the motivating assumption for this paper, that extending Johnson’s (2006a) affordable housing model to provide more realism in development characteristics such as scale effects and social equity concerns results in tractable models that provide a wider range of choices for planners.

Model Extensions

Stronger policy insight for these planning models is clearly dependent on dollar-valued impacts of alternative affordable housing strategies that are based on explicit economic models of consumer behavior combined with forecasting models, a broader sample of provider data than has been available to date and provider-defined values for key structural parameters. It will then be possible to formally
validate model findings and infer the potential benefits associated with use of a formal math
programming-based planning model.

The planning models in this paper rely on a number of common assumptions. First, the
development choices for the affordable housing provider are limited to new construction and renovation.
Second, all sites on which affordable housing might be provided are available and ready for development.
Third, the planning problem takes place over a single period, in which all data are known with certainty.
By relaxing these assumptions we generate interesting and challenging planning models that alternatively:
address additional developer strategies such as land-banking and rapid property turnover, allow
developers to choose among multiple property acquisition strategies, and address housing development
over multiple periods under risk and uncertainty.

VII. Conclusion

This paper has presented a collection of realistic models to generate alternative housing strategies.
Housing has special characteristics that distinguish it from other facilities: customers may travel to
housing from various distances and do so only periodically, and housing relies on location in residential
contexts. Thus, we adapt the capacitated fixed-charge location problem by replacing distance between
“facilities” and “customers” with variable net social impacts. We have argued that affordable housing has
multiple dimensions of impacts and design characteristics. The models in this paper reflect for the first
time real-world perspectives on scale impacts of affordable housing development, as well as conflicting
stakeholder and policymaker concerns regarding potential social outcomes of such development.

Preliminary computational results for the affordable housing location problem with scale effects
for dollar-valued impacts (AFFLOC) and alternative equity objective formulations are encouraging.
There appear to be tradeoffs between computation time, theoretical support and variation in model
outcome values for model using the minimax equity measure, the segregation index equity measure and
the employment variance equity measure. Nondominated solutions generated by AFFLOC using the
minimax equity measure show the greatest variety in housing development sizes and consume fairly large
amounts computational resources without meeting a user-defined time limit. However, the minimax equity measure does not have a strong basis in current housing policy. Nondominated solutions generated by AFFLOC using the segregation index equity measure are virtually indistinguishable from each other in objective space and according to the distribution of housing development sizes, though there is significant spatial variation in solutions. Runtimes for this model variant are trivial. AFFLOC formulated using the employment equity measure, which is motivated by research on the spatial mismatch hypothesis, requires significant computational resources. However, nondominated solutions show plausible spatial variation according to levels of entry-level employment. Also, candidate compromise solutions perform very well in both objective dimensions and show modest variation in housing development sizes.

Extensions to this research include greater precision in dollar-valued impacts of affordable housing and model structural parameters to enable these models to be validated against historical decisions, as well as a collection of related affordable housing planning models that address a number of real-life complications. These include: a wider range of development decisions regarding available land parcels, the choice of when and how many parcels for housing redevelopment to acquire to account for limited funding, and incorporation of uncertainty regarding values of structural parameters describing housing units and housing markets over multiple periods.

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References


  Toronto: Rotman School of Management, University of Toronto.


### Tables and Figures

<table>
<thead>
<tr>
<th>Attribute</th>
<th>Dollar-Valued Category</th>
<th>Qualitative or Quantitative</th>
<th>Range of Values</th>
<th>Contribution to Scale Effects</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Housing Unit</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1. Tenure type</td>
<td>Variable benefits, fixed costs</td>
<td>Qualitative</td>
<td>Renter-occupied; owner-occupied</td>
<td>Owner-occupied units could increase social benefits at a greater rate than renter-occupied units due to possibility for wealth accumulation and neighborhood stabilization.</td>
</tr>
<tr>
<td>2. Subsidy type</td>
<td>Variable benefits, fixed costs</td>
<td>Qualitative</td>
<td>Tenant-based; project-based</td>
<td>Units with project-based subsidies may require greater fixed provision costs, but units with tenant-based subsidies may require greater periodic investments.</td>
</tr>
<tr>
<td>3. Size</td>
<td>Variable benefits, fixed costs</td>
<td>Quantitative</td>
<td>0, 1, … bedrooms; square feet</td>
<td>Social net benefits might increase nonlinearly in the size of a housing unit, since large low-income families are most in need of affordable housing and most at risk for deleterious outcomes in the absence of such housing.</td>
</tr>
<tr>
<td><strong>Housing Structure</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>4. Provision method</td>
<td>Fixed costs</td>
<td>Qualitative</td>
<td>Construction; purchase; rehabilitation</td>
<td>Purchase of existing units is likely to generate scale economies as compared to renovation or new construction, especially in older housing markets.</td>
</tr>
<tr>
<td>5. Residential type</td>
<td>Fixed costs</td>
<td>Qualitative</td>
<td>Single-family; multi-family</td>
<td>Scale economies might occur more quickly for higher-density multifamily units than attached single-family units.</td>
</tr>
<tr>
<td>6. Size</td>
<td>Variable benefits, fixed costs</td>
<td>Quantitative</td>
<td>1, 2, … units; square feet</td>
<td>Scale economies might occur more quickly for larger-sized structures than smaller-sized structures, up to a threshold</td>
</tr>
<tr>
<td>7. Subsidy type</td>
<td>Fixed costs</td>
<td>Qualitative</td>
<td>Tax credits; grants, below-market rate loans</td>
<td>If tax credits are generally larger than other subsidy mechanisms, they may enable simultaneous development of more units, and thus scale economies.</td>
</tr>
<tr>
<td>8. Subsidy level</td>
<td>Variable costs</td>
<td>Quantitative</td>
<td>Total subsidy for structure</td>
<td>Subsidy levels are likely to increase at a decreasing rate in the number of units due to economies in provision costs.</td>
</tr>
</tbody>
</table>

[Table 1: Analysis of Housing Initiative Attributes]

[Figure 1: Net Social Benefits of a Housing Development with Scale Effects]
[Figure 2: Pareto Frontier, Model AFFLOC with Center Equity Measure]

[Figure 3: Development Sizes, Model AFFLOC with Center Equity Measure]

[Figure 4: Spatial Distribution, Model AFFLOC with Center Equity Measure]
[Figure 5: Pareto Frontier, Model AFFLOC with Segregation Equity Measure]

[Figure 6: Pareto Frontier, Model AFFLOC with Employment Variance Equity Measure]

[Figure 7: Development Sizes, Model AFFLOC with Employment Variance Equity Measure]