On the capacity of isolated, curbside bus stops

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Abstract

The maximal rates that buses can discharge from bus stops are examined. Models were developed to estimate these capacities for curbside stops that are isolated from the effects of traffic signals. The models account for key features of the stops, including their target service levels assigned to them by a transit agency. Among other things, the models predict that adding bus berths to a stop can sometimes return disproportionally high gains in capacity. This and other of our findings are at odds with information furnished in professional handbooks.

1. Introduction

While serving passengers at a busy stop, buses can interact in ways that limit their discharge flows. This can degrade the bus system’s overall service quality (Fernandez, 2010; Fernandez and Planzer, 2002; Gibson et al., 1989).

The present paper explores the bus discharge flows that can be achieved at stops where buses dwell curbside to load and unload passengers. We will examine stops that are isolated from the influences of traffic signals and other bus stops; where sufficient space exists for storing the bus queues that can form immediately upstream of the stops; where bus movements in and around the stops are not affected by other (e.g. car) traffic; and where bus overtaking maneuvers are prohibited, both within any bus queues immediately upstream, and at the stops themselves, should multiple berths (i.e. bus loading areas) exist there.1

The rates that buses can discharge from stops of this kind depend in part on the target service level chosen by the transit agency. In this paper, we use a metric of service level called the failure rate, $FR$, defined as the probability that a bus arriving to a stop is temporarily blocked from using it by another bus. Though other service metrics (e.g. average bus waiting time) are possible, $FR$ is the metric featured in professional handbooks (e.g. TRB, 2000, 2003). Intuitively, the bus discharge flow increases as $FR$ increases, and is highest when a bus queue is always present at the stop’s entrance, i.e. when $FR = 1$.2

In light of this influence, we shall define bus-stop capacity as the maximal rate that buses can discharge from a stop for a specified threshold of $FR$. This definition is common in the literature (see again TRB, 2000, 2003). Our findings, on the other hand, are largely at odds with earlier publications, as we shall see. We shall arrive at these findings by developing (and evaluating) models that predict bus-stop capacities as functions of not only $FR$, but also bus arrival process and bus service time distribution.

A review of earlier work is furnished in the following section. Present findings in regard to stops with only one berth are provided in Section 3. Findings on multi-berth stops are in Section 4. Practical implications are discussed in Section 5.

1 Cities often enact this prohibition because an overtaking bus can disrupt car traffic in the adjacent lane(s).
2 If buses were controlled so that their arrival headways and service time at a stop were perfectly coordinated, the stop could, in theory, always be occupied without queues forming. The $FR$ in this idealized (and unrealistic) case would therefore be zero, though the bus discharge flow would be high.
2. Literature review

The *Highway Capacity Manual* (TRB, 2000) reports that the capacity of a single-berth stop is inversely proportional to the sum of (i) the bus’ average service time; and (ii) a second term that accounts for both the variation in this service time and the FR.\(^3\) With this latter term, a stop’s capacity increases with increasing FR, but only to a point. Curiously, the formula in the *Highway Capacity Manual* (henceforth HCM) predicts that capacity is maximal when FR reaches 0.5. Intuition, on the other hand, tells us that single-berth capacity is maximal when a bus queue always persists upstream; i.e. when FR is 1. Of further concern, the current edition of the HCM omits any discussion on the influence of the bus arrival process on stop capacity.\(^4\)

For a multi-berth stop, the HCM takes capacity to be the product of the single-berth capacity and the number of “effective” berths. The HCM furnishes values for this latter term that result in steadily diminishing returns in capacity, meaning that each new berth that is added to a stop will return less than a proportional increase in the stop’s capacity (see Table 27-12 of the TRB, 2000). Presumably, this is to account for the disruptive bus interactions that can occur at multi-berth stops (see our discussion of the “blocking effect” in Section 4.1). However, the inefficiencies brought with each added berth are assumed in the HCM to be independent of all other factors, including: FR, bus arrival process, and service time variation.

Much of the above is at odds with our present findings (see Sections 3 and 4). What thus appear to be shortcomings of the HCM take on greater significance because they are repeated in the *Transit Capacity and Quality of Service Manual* (TRB, 2003). This latter handbook will reportedly supplant discussion of transit systems in future editions of the HCM. The same ideas, moreover, have found their way into the *Transportation Planning Handbook* (ITE, 1999).

Critiques of these capacity formulas already appear in the literature. Gibson et al. (1989), for example, argues that the complex stochastic processes at real bus stops limit the usefulness of HCM formulas. Fernandez and Planzer (2002) reports that the formulas tend to under-predict field-measured estimates of stop capacity. These findings are useful in that they highlight certain influences on bus-stop capacity. Yet, they do little to quantify these influences. Similarly, studies to increase the capacity of a multi-berth stop by either dispatching buses in certain ways (Gardner et al., 1991; Szász et al., 1978), or by reconfiguring the stop’s geometry (Gibson et al., 1989; St. Jacques and Levinson, 1997; etc.) offer only limited insights into cause and effect. The same is true of past efforts to estimate the parameter values for describing bus arrival processes (Danas, 1980; Fernandez, 2001; Ge, 2006; Kohler, 1991) and service time distributions (Ge, 2006; St. Jacques and Levinson, 1997).

3. Single-berth stops

It will be assumed that bus stops operate in the steady-state, such that the arrival process and the service time distribution are both time-invariant, and that the long-run average bus arrival rate never exceeds the stop’s capacity when FR is 1. In this steady-state, the average bus inflow to the stop always equals the average outflow.

Although some empirical studies show that bus arrivals at stops follow a Poisson process (Danas, 1980; Ge, 2006; Kohler, 1991), other studies (e.g. Fernandez, 2001) argue that this is not always the case. To simplify our analysis and highlight the findings, we start by assuming two special cases in regard to the bus arrival process: Poisson arrivals (in Section 3.1), as can occur when the stop serves a single route with buses that are rigidly controlled. Finally, Section 3.3 examines the case of a more general bus arrival pattern. Capacity formulas will be furnished for each of these three cases.

3.1. Poisson bus arrivals

In the steady-state, Poisson bus arrivals to a stop satisfy the Poisson Arrivals See Time Averages (PASTA) property; see Wolff (1982). This implies that FR is equal to the fraction of time that the stop’s single berth is utilized. This utilization fraction is the ratio of bus inflow, \(\lambda\), to the single-berth stop’s maximal service rate (i.e. the inverse of the average time that each bus spends serving passenger boarding and alighting movements). We denote this maximal service rate as \(\mu\). Thus, for \(\lambda \leq \mu\),

\[
\frac{\lambda}{\mu} = FR
\]

Since \(\lambda\) can be viewed equivalently as the stop’s capacity for a specified FR; and since \(\mu\) is the stop’s output flow when FR = 1; the ratio \(\frac{\lambda}{\mu}\) will henceforth be termed the normalized capacity.

As per intuition, (1) shows that single-berth capacity is maximal when FR = 1. It further shows that for Poisson bus arrivals, capacity is independent of the variation in bus service time (for boarding and alighting movements). This independence turns out not to hold in general, however, as we shall see next.

\(^3\) The second term involves both: the one-tail standard normal variate corresponding to FR; and the coefficient of variation of bus service time (see Eq. 27-5 of TRB, 2000).

\(^4\) Although an earlier edition of the HCM includes a multiplicative adjustment factor that reportedly accounts for variations in bus arrival headway, the factor seems instead to account for FR (see Eq. 12-7 and Table 12-17 of TRB, 1985).
3.2. Uniform bus arrivals

Assume now that the bus arrival headways are deterministic and equal. Further assume that bus service time follows an Erlang-$k$ distribution, which is a more general distribution than the commonly-used exponential distribution (and has been observed in Ge, 2006 to be suitable at some stops.) For this present case, our model does not have a closed-form solution. An analytical model that can be solved numerically is derived in Appendix A. A simple, closed-form approximation to the solution of this model is found to be:

$$k \approx \frac{FR}{CS_{1.58 + 0.63CS}}$$

where $CS$ is the coefficient of variation in bus service time.

Eq. (2) came by fitting a curve to our numerical solutions over the range of $CS \in [0, 1]$, since this is consistent with the range of $CS$ observed in the literature (St. Jacques and Levinson, 1997). The result satisfies intuitive boundary conditions for the relation between $FR$ and $k$. The inclusion of $CS$ in (2) is logical, since $k = C_{S}^{2}$ for Erlang distributions, and this shows how stop capacity for the case of uniform bus arrivals depends on the coefficient of variation in bus service time as well as on $FR$.

To explore matters more deeply, relations generated from (2) are shown with solid curves in Fig. 1 for $CS = 0.1$, $0.5$, and $1$. These curves collectively reveal that, for uniform arrivals and for $0 < FR < 1$, capacity increases as the coefficient of variation in bus service time diminishes. The curves further show that the maximal capacity of the stop (when $FR = 1$) is the same for all $CS$. The case of $CS = 0$ corresponds to the perfect coordination of bus arrivals and bus service time, as previously discussed in Footnote 2, such that $FR = 0$. The curve in this idealized case therefore reduces to a point, also as shown in Fig. 1.

The relation for Poisson bus arrivals revealed in (1) is shown in Fig. 1 as well; see the dashed line. Comparing this dashed line against the solid curves reveals that for $0 < FR < 1$, capacity also increases with diminishing variation in bus headway. (We can see this because the coefficient of variation is 0 and 1 for uniform and Poisson bus arrivals, respectively.)

3.3. General bus arrivals

We continue to model bus service time as above, and now use the Erlang-$j$ distribution to describe a more general distribution for bus headways. A numerical solution was derived in similar fashion to the case of uniform bus arrivals, for which an approximation is found to be:

$$\lambda \approx \frac{FR^{0.44 + 0.15CS - 0.2W_{C_{S}}}}{W_{S}}$$

where $C_{H}$ is the coefficient of variation of bus arrival headway.

From (3) we see that stop capacity is influenced by service time and headway variations. Readers can verify that reductions in the coefficient of variation for either of these factors will increase a stop’s capacity when $0 < FR < 1$, and $0 \leq C_{H}, C_{S} \leq 1$; e.g. one can fix either $C_{H}$ or $C_{S}$ and obtain curves that are qualitatively similar (in their shapes and their relative positions) to the solid curves in Fig. 1.\footnote{These conditions are: (i) $\lambda = 0$ if $FR = 0$ and $C_{S} > 0$; and (ii) $\lambda = 1$ if $FR = 1$ or if $C_{S} = FR = 0$.}

\footnote{In addition to satisfying the conditions in Footnote 5, Eq. (3) reduces to (1) for the case of Poisson bus arrivals where $C_{H} = 1$. As an aside, analysis shows that (3) produces significantly lower capacities as compared with the formulas of the HCM (TRB, 2000).}

Fig. 1. Normalized capacity versus $FR$ for single-berth stops; comparisons between Poisson and uniform bus arrivals.
4. Multi-berth stops

Two competing effects, which we term the “blocking” and the “berth pooling” effects, are found to influence the capacity of multi-berth stops, as explained in Section 4.1. The returns in capacity with each berth added (henceforth referred to as “returns in capacity”) are studied for two limiting cases that isolate the above effects and for a third, more general case, all in Section 4.2. Further findings come by examining how returns in capacity are influenced by coefficients of variation in bus service time and bus headway, as shown in Section 4.3. For all these analyses, we will assume that the distribution of an individual bus’ service time (to load and unload passengers) is independent of the stop’s number of berths.

4.1. Two competing effects

Discussion begins with the blocking effect. A bus can enter a stop only when its upstream-most berth is open. (At this time, the entering bus proceeds as far as possible until encountering the end of the stop or a dwelling bus; and the entering bus will then dwell at the downstream-most available berth for its entire time in the stop.) Similarly, a bus can discharge from a stop only after all buses that were previously dwelling at that stop’s downstream berths have departed. This blocking effect for entering and exiting a stop tends to diminish the stop’s returns in capacity to added berths. The effect diminishes, however, when the load intensity per-berth, \( R \), approaches 0, where \( R = \frac{k}{c} \) and \( c \) is the number of berths at the stop.

We illustrate the second effect, berth pooling, with the following example. Consider two independent, single-berth stops, each with equal bus arrival rate, \( \lambda \), as shown on the left side of Fig. 2. (Dashed boxes in this figure denote berths, and shaded rectangles denote buses.) If we ignore the blocking effect, the fluctuations in bus arrivals would be better served by pooling the two berths into a single, double-berth stop, as shown on the right side of Fig. 2. Thus for the same total bus arrival rate (2\( \lambda \) for both the left and right sides in the figure), this berth pooling effect means that the double-berth stop would enjoy a lower FR than would the two single-berth stops; i.e. the double-berth stop would have a higher capacity for a given FR. Berth pooling tends to improve the stop’s returns in capacity to added berths. The effect diminishes, however, when \( R \) approaches its maximum, meaning when the input flow, \( \lambda \), approaches the stop’s maximal capacity (see Eq. (4)).

The above effects are countervailing: as \( R \) approaches 0 or its maximum, one effect diminishes while the other dominates.\(^7\) We will therefore isolate the two effects by examining multi-berth stops under the two limiting cases for \( R \).

4.2. Returns in capacity

We next explore the returns in capacity (i) when \( R \) is maximal; (ii) when \( R = 0 \); and (iii) for the general case when \( R \) falls between these limits.

4.2.1. Limiting case when \( R \) is maximal

In this case, queued buses enter a stop in platoons of size \( c \), and the time required to serve a platoon is the maximal bus service time across the platoon. The stop’s maximal capacity, \( Q(c) \), is therefore:

\[
Q(c) = \frac{c}{E[T(c)]} = \frac{c}{\int_{t=0}^{\infty} (1 - F_s(t))^c dt}
\]  

(4)

where \( E[T(c)] \) is the expected value of the platoon service time; and \( F_s(t) \) is the cumulative distribution function of the individual bus service time. The derivation of (4) is furnished in Appendix B. Intuitively, the bus arrival pattern (to the rear of the queue) does not influence capacity in this limiting case.

The average capacity per berth, \( \frac{1}{c} \), decreases with added berths, since \( E[T(c)] \) increases with \( c \). Thus from the first equality in (4), we see how the blocking effect can create decreasing returns in capacity.

\(^7\) As per Footnote 2, an exception can occur under perfect coordination; i.e. when platoons of \( c \) buses arrive at uniform intervals and the service time is constant. In this case, neither blocking nor berth pooling take effect and the FR is always zero.
4.2.2. Limiting case of $R \to 0$

Computer simulation is used next to explore stop capacity under this second limiting case. The logic of our simulation model is described in Appendix C. For the analysis to follow, bus service time is assumed to follow the gamma distribution (a generalization of the Erlang distribution) with $C_s = 0.6$, as recommended by St. Jacques and Levinson (1997). Bus arrivals are assumed to follow a Poisson process, as if the stop were used by multiple bus lines. Simulations of other bus arrival patterns and service time distributions yield qualitatively similar results.

The curves in Fig. 3 display the normalized incremental change in stop capacity achieved for each added berth, $\Delta \lambda/\mu$, for the first through the sixth berth. These curves are shown for near-zero values of $FR$, since it is the assumed metric of interest and is a reasonable proxy for $R$. (Note that $FR$ approaches zero when $R$ does so, and that the maximal value of one coincides with the maximal value of the other.) The curves reveal that $\Delta \lambda/\mu$ increases with each additional berth; i.e. that added berths bring increasing returns in capacity.

Fig. 3. Increasing returns in capacity caused by berth pooling effect.

Fig. 4. Normalized stop capacity and incremental change in capacity versus $FR$ for multi-berth stops with Poisson bus arrivals and gamma-distributed service time ($C_s = 0.6$).
Although $R$ and $FR$ might seldom approach zero in an urban setting, the finding calls into question what handbooks have to say on the subject; i.e. the implication that added berths bring decreasing returns in capacity does not hold in general. More interesting evidence in this regard comes next.

4.2.3. General case with intermediate values of $R$

We now use our simulation model (see again Appendix C) to explore bus-stop capacities when $FR$ is between 0 and its maximum. Once again, bus arrivals are assumed to be Poisson, and service time gamma-distributed with $CS = 0.6$.

The curves in Fig. 4a display the $Dk$ for the first through the sixth berth. These too are shown as functions of $FR$, our chosen service metric and proxy for $R$. The curves reveal how the countervailing effects of blocking and berth pooling produce mixed results in terms of the capacities returned by adding berths to a stop.

When $FR$ is small (but not approaching zero), additional berths can produce increasing returns in capacity, thanks to the berth pooling effect. For example, the figure shows that when $FR \approx 0.1$, adding a second berth brings increasing returns. (Note that when $FR \approx 0.1$, the curve for the second berth lies above that for the first.) This favorable trend does not continue, however. Note, for example, now the curve for the third berth lies below that for the second when $FR \approx 0.1$.

Toward the other extreme (e.g. when $FR \approx 0.7$), the curves reveal that added berths produce diminishing returns in capacity. This is because the blocking effect tends to dominate.

These findings are logical in light of what was unveiled for the two limiting cases. Yet, our finding that returns in capacity vary with $FR$ or $R$ runs counter to the HCM’s suggestion in this regard; i.e. using a single set of numbers for “effective berths” evidently does not suffice for all operating environments.

A graph like Fig. 4a can be used in a number of practical pursuits. The same is true for variants, like the curves of $FR$ versus normalized capacity shown in Fig. 4b. More will be said on these matters in Section 5.

4.3. Variations in service time and headway

Having explored the influences of $FR$ and $R$, we now examine how the returns in capacity are influenced by the coefficients of variation in bus service time and bus headway. Simulation is again used to this end.

4.3.1. Bus service time

We continue to assume that bus arrivals are Poisson and that service time is gamma-distributed. Now, however, capacities will be explored for the range of $CS \in [0, 1]$.

Fig. 5a displays effects of $CS$ on the $Dk$ for the first through the sixth berth when $FR = 0.15$. Note from the figure that increased returns in capacity come by adding a second berth to a stop (i.e. the curve for the second berth lies above that for the first). This is again thanks to the pooling effect at low $FR$. Further note that the curves for the second through the sixth berth exhibit downward slopes. This reveals an inverse influence of $CS$ on the returns in capacity. Additionally, the downward sloping curves for $c = 2–6$ in Fig. 5b reveal how $CS$ exerts an inverse influence on stop capacity itself. These inverse influences become more dramatic as $FR$ increases. To illustrate, the above analysis is repeated, but for $FR = 1$. Results are displayed in Fig. 6a and b.

4.3.2. Bus headway

To explore how variations in bus arrival headway affect things, we will assume that: $FR = 0.15$; bus service time is gamma-distributed with $CS = 0.6$; and bus headway is also gamma-distributed with a coefficient of variation, $CH$, ranging from 0 to 1.

![Fig. 5. Normalized capacity and incremental change in capacity versus $CS$ for multi-berth stops with $FR = 0.15$.](image-url)
The curves in Fig. 7a show that the first berth is relatively sensitive to $CH$; i.e. when $c = 1$, the $Dk$ diminishes precipitously with increasing $CH$. As a result, the $Dk$ for the second through even the sixth berth is greater than that achieved by the first berth when $CH$ is sufficiently high. For example, we see that adding a second berth to a stop produces an increased return in capacity once $CH$ comfortably exceeds 0.6. Once again, however, we find that a stop's capacity for any $c$ diminishes as $CH$ grows large; see Fig. 7b. The above influences are found to disappear as $FR$ approaches 1.

5. Conclusions

The models presented in this paper account for key influences on the capacities of isolated, curbside bus stops. They do so in ways that are more complete than what has been offered by formulas in well-known handbooks. Through this more complete accounting come insights. The insights have practical implications.

For example, the models predict that variations in bus service time tend to diminish stop capacity, both for single- and multi-berth stops. (See Figs. 1, 5b and 6b, and recall that an exception to this occurs when buses arrive at a single-berth stop as a Poisson process.) This finding speaks to the value of reducing service-time variations via the improved management of passenger boarding and alighting. Means of doing this might include the use of wider bus doors, improved loading platforms and off-board fare collection. Of course, these measures could also help reduce the average service time, and this too would favorably affect bus-stop capacity.

In contrast to formulas in professional handbooks, the present models also account for the effects of the bus arrival process at a stop. They predict that variations in bus headway can diminish stop capacity (Fig. 7b), but can in some instances favorably affect the returns in capacity brought by a second through even a sixth berth relative to the returns from a single berth (Fig. 7a). When the variation in headway is high and the $FR$ is low, adding a second berth to a single-berth stop can bring an increased return in capacity (Figs. 4a and 7a). Knowledge of these cause and effect relations can be useful when choosing the number of berths to be deployed at a curbside stop.
To further illustrate the practical utility of our models, we ask the reader to refer again to Fig. 4b. It displays relations between FR and normalized capacity for stops that range in size from 1 to 6 berths. Note how the curves in this figure can be used to determine the number of berths needed to achieve targets for FR and stop capacity. Or, they can be used to estimate FR given bus arrival rate and a specified number of berths. The figure can also help determine when it can be advantageous to split a single stop with many berths into multiple adjacent stops. For example, the reader can use Fig. 4b to verify that, for a FR of 0.5, splitting a 4-berth stop into two 2-berth stops could increase capacity by nearly 15%. (That capacity is increased by splitting the stop is clearly evident in Fig. 4a, since at FR = 0.5, the \( \frac{1}{2} \) for the third and fourth berths are lower than for the first and second berths.) Admittedly, this prediction assumes certain idealizations; e.g. that both the bus arrival processes and the service time distributions are comparable across the 2-berth stops; and that buses bound for one of these stops do not impede buses bound for the other.

To be sure, all of our present models are idealized, particularly since they apply to isolated stops operating in steady-state. Yet in our view, these models represent a step toward better understanding bus-stop operation. Work is ongoing in regard to stops: that are not isolated, but are instead affected by traffic signals and other bus stops; that have limited space for storing bus queues; and that allow bus overtaking. In the mean time, one may still use our models to develop graphs that are similar to those shown here, but that are tailored to local conditions for target FR, variations in service time and headway, and so on.

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Appendix A. Analytical solution to a single-berth stop with uniform bus arrivals and Erlang-k service time (in Section 3.2)

Here we furnish a solution by applying a more general result given by Gross et al. (2008) for a queueing system with bus headway and service time following generalized Erlang distributions, \( GE_j/GE_k/1 \), where \( GE_j \) and \( GE_k \) are the distributions of bus headway and bus service time, respectively.\(^8\) This general result is:

\[
\mathcal{W}_q(s) = \frac{\Pi_{m=1}^{k} \left( -\frac{s_m}{s} \right)(s + \mu_m)}{s\Pi_{m=1}^{k}(s - s_m)}
\]

(A.1)

where \( \mathcal{W}_q(s) \) is the Laplace–Stieltjes transform of the cumulative distribution function (CDF) of bus waiting time before entering a berth; \( \mu_m(m = 1, \ldots, k) \) is the rate of the \( m \)th exponential component of the \( GE_k \) distribution; and \( s_m(m = 1, \ldots) \) is the \( m \)th complex root with a negative real part of the following equation with argument \( s \):

\[
\Pi_{m=1}^{l} \lambda_m \Pi_{m=1}^{k} \mu_m - \Pi_{m=1}^{l} (\lambda_m - s) \Pi_{m=1}^{k} (\mu_m + s) = 0
\]

(A.2)

Eq. (A.2) is also given in Gross et al. (2008). The \( \lambda_m(m = 1, \ldots, j) \) is the rate of the \( m \)th exponential component of the \( GE_j \) distribution.

Since the means of the headway and the service time are \( \frac{1}{\lambda_m} \) and \( \frac{1}{\mu_m} \), respectively, we set \( \lambda_m(m = 1, \ldots, j) = j \lambda, \mu_m(m = 1, \ldots, k) = k \mu = k \) so that the bus headway and the service time are Erlang-\( j \) and Erlang-\( k \) distributed, respectively; and so that the bus arrival rate and the service rate are \( \lambda \) and \( \mu = 1 \), respectively. Given that when \( j \) approaches infinity, the limit of the Erlang-\( j \) distribution is a deterministic value, we let \( j \to \infty \), so that the headway becomes constant. Then (A.2) becomes

\[
\left( \frac{k}{1+1} \right)^k = e^x.
\]

Let \( R = \lambda/\mu = \lambda \), such that the solution of the above equation is:

\[
s_m = -kR \cdot \text{LambertW} \left( -\frac{1}{R} e^{+\frac{2m-1}{R}} \right) - 1, (m = 1, \ldots, k)
\]

(A.3)

where function \( \text{LambertW}(\cdot) \) is the inverse function of \( f(w) = we^w \), which is multi-valued in the field of complex numbers, and has no closed-form expression; and \( i \) is the imaginary unit.

By picking up the roots of \( s_m \)'s with negative real parts, plugging them into (A.1), and then taking a partial-fraction expansion, we obtain:

\[
\mathcal{W}_q(s) = \frac{\Pi_{m=1}^{k} \left( -s_m \left( \frac{k}{2} + 1 \right) \right)(s + \mu_m)}{s\Pi_{m=1}^{k}(s - s_m)} - \frac{1}{s} - \sum_{m=1}^{k} \frac{\beta_m}{s - s_m}
\]

(A.4)

\(^8\) A generalized Erlang random variable is the convolution of independent but not necessarily identical exponential random variables. Here a bus headway can be expressed as the sum of \( j \) exponential components that are independent but may not be identical; and a bus service time can be expressed as the sum of \( k \) such components.
where $\beta_m$ are constant coefficients to be determined by:

$$\beta_m = \prod_{n=1}^{k} \left( -\frac{S_n}{S_m - S_n} \right) \left( \frac{S_m}{k} + 1 \right)^k$$

(A.5)

By applying the inverse Laplace transform on (A.4), we obtain the CDF of the bus waiting time: $W_q(t) = 1 - \sum_{m=1}^{k} \beta_m e^{\alpha t}$

Therefore the failure rate becomes

$$FR = 1 - W_q(0) = \sum_{m=1}^{k} \beta_m$$

(A.6)

For any given $k = C_5^2$, the last term of (A.6) is a function of $R = C_6$. Thus we find the relation between $FR$ and $C_6$. The results can be obtained numerically.

Appendix B. Derivation of Eq. (4) in Section 4.2.1

For a fixed number of berths, $c$, let $T(c) = \max\{S_j\}_j (j = 1, 2, \ldots, c)$ be the platoon service time, where $\{S_j\}$ is the service time of the $j$th bus in the platoon. All $\{S_j\}$'s are independent, identically distributed random variables subject to the CDF of $F_j(t)$. Let $F_{T(c)}(t)$ be the CDF of $T(c)$. Thus we have:

$$F_{T(c)}(t) = P\{T(c) \leq t\} = P\{\tilde{S}_j \leq t, j = 1, 2, \ldots, c\} = \prod_{j=1}^{c} P\{\tilde{S}_j \leq t\} = \prod_{j=1}^{c} F_j(t) = (F_j(t))^c$$

From the identity $E[T(c)] = \int_{t=0}^{\infty} (1 - F_{T(c)}(t))dt$, we have:

$$E[T(c)] = \int_{t=0}^{\infty} (1 - (F_j(t))^c)dt.$$

Appendix C. Simulation algorithm for the multi-berth stops analyzed in Sections 4.2 and 4.3

First we introduce the following notation used in our simulation model:

$H_i$ – Headway (in minutes) between the arrivals of $Bus_{i-1}$ and $Bus_i$, and $H_1$ is the system idle time before the first bus arrives;

$S_i$ – Service time (in minutes) of $Bus_i$, not including the time that $Bus_i$ waits to depart the stop after it has finished serving passengers;

$P_i$ – The position (number) of the berth where $Bus_i$ dwells to serve passengers; where berths are numbered 1, 2, $\ldots$, $c$ from the downstream to the upstream berth;

$W_{iq}$ – Waiting time in the queue (in minutes) of $Bus_i$ before it enters the stop;

$W_{ib}$ – Waiting time in the berth (in minutes) of $Bus_i$ after its service is finished; and

$F_i$ – Indicator that takes 1 if $Bus_i$ fails to enter the berth immediately upon its arrival to the stop, and 0 otherwise.

The dynamic equations describing our simulation model are:

For each $i = 1, 2, \ldots$,

$$W_{i+1,q} = \begin{cases} \max \{0, W_{iq} + S_i + W_{ib} - H_{i+1}\}, & \text{if } P_i = c \\ \max \{0, W_{iq} - H_{i+1}\}, & \text{otherwise}; \end{cases}$$

$$P_{i+1} = \begin{cases} 1, & \text{if } W_{iq} + S_i + W_{ib} - H_{i+1} - W_{i+1,q} \leq 0 \\ P_i + 1, & \text{otherwise}; \end{cases}$$

$$W_{i+1,b} = \max\{0, W_{iq} + S_i + W_{ib} - H_{i+1} - W_{i+1,q} - S_{i+1}\};$$

$$F_{i+1} = \begin{cases} 1, & \text{if } W_{i+1,q} > 0 \\ 0, & \text{if } W_{i+1,q} = 0. \end{cases}$$

The $H_i$ and $S_i (i = 1, 2, 3, \ldots)$ are inputs to the simulation. We assume that $H_i$ follows a gamma distribution with mean $\frac{1}{\lambda}$ and coefficient of variation $C_{\lambda}$. (For Poisson bus arrivals, $C_{\lambda} = 1$ and $H_i$ is exponentially distributed.) We further assume that $S_i$ follows another gamma distribution with mean $\frac{1}{\mu}$ and coefficient of variation $C_\mu$. The simulation starts from an initial state in which the stop is empty (i.e. $W_{1,q} = W_{1,b} = F_1 = 0$ and $P_1 = 1$) and ends at the same state to diminish stochastic error. The resulting performance measure $FR$ is the long-term average of $F_i$. 

References